Dancing on the Saddles: A Geometric Framework for Stochastic Equilibrium Dynamics[1]

Online Appendix

Contents

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A Sequential formulation of a canonical RBC

A representative household with temporal log utility is considered. Given initial condition (a_0, A_0), the household maximizes life-time utility under stochastic aggregate TFP A_t, which is subject to a budget constraint as elaborated on below:

$$\max_{\{c_\tau(A^{(\tau)}),a_{\tau+1}(A^{(\tau)})\}_{\tau=0}^{\infty}} \mathbb{E}_0 \sum_{\tau=0}^{\infty} \beta^\tau log(c_\tau(A^{(\tau)})) \tag{1}$$

$$\text{s.t. } c_\tau(A^{(\tau)}) + a_{\tau+1}(A^{(\tau)}) = a_\tau(A^{(\tau-1)})(1 + r(A^{(\tau)})) + w(A^{(\tau)}), \quad \text{for } \forall\tau, \forall A_t \tag{2}$$

$$a_{\tau+1}(A^{(\tau)}) \geq -\overline{a}, \quad \text{for } \forall\tau \tag{3}$$

where superscript τ inside a bracket denotes history of a variable up to period τ; $-\overline{a}$ is the natural borrowing limit to preempt Ponzi scheme. Labor supply is exogenously fixed at unity. I consider the following competitive factor prices given CRS Cobb-Douglas production function:

$$r(A^{(\tau)}) = A_t\alpha(K(A^{(\tau)}))^{(\alpha-1)} - \delta \tag{4}$$

$$w(A^{(\tau)}) = A_t(1 - \alpha)(K(A^{(\tau)}))^{\alpha}, \tag{5}$$

K is capital stock, that satisfies $K(A^{(\tau)}) = a(A^{(\tau)})$ in equilibrium. With the regularity conditions given in Stokey et al. (1989), this sequential formulation yields the same optimality conditions as the recursive form in the main text.

B Individual conditional saddles in Aiyagari (1994)

I define individual-level conditional saddle path in the heterogeneous-household economy without aggregate uncertainty ($A = A' = 1$). The conditional saddle is defined for the SRCE as in Aiyagari (1994).

Definition 1 (Individual reachable equilibrium set)**.**
Fix a stationary RCE of the Aiyagari (1994) economy, which delivers an individual asset law of motion

$$g_a : \mathbb{R}_+ \times \mathcal{Z} \to \mathbb{R}_+,$$

where $\mathcal{Z}$ is a finite Markov set (e.g. $\mathcal{Z} = \{u, e\}$) with transition matrix Π. Fix an initial condition $(a_0, z_0) \in \mathbb{R}_+ \times \mathcal{Z}$.

The individual reachable equilibrium set $\mathcal{R}^{\text{ind}}(a_0, z_0) \subseteq \mathbb{R}_+ \times \mathcal{Z}$ *is the smallest set* *containing* (a_0, z_0) *that is closed under all feasible one-step equilibrium transitions:*

$$(a, z) \in \mathcal{R}^{\text{ind}}(a_0, z_0) \quad \textit{and} \quad \Pi_{zz'} > 0 \quad \Longrightarrow \quad \big(g_a(a, z'),\, z'\big) \in \mathcal{R}^{\text{ind}}(a_0, z_0).$$

Definition 2 (Individual conditional saddle)**.**
Given (a_0, z_0) *and* $\mathcal{R}^{\text{ind}}(a_0, z_0)$*, for each* $z \in \mathcal{Z}$ *define the* z*-slice*

$$\mathcal{R}^{\text{ind}}_z(a_0, z_0) := \{a \in \mathbb{R}_+ : \ (a, z) \in \mathcal{R}^{\text{ind}}(a_0, z_0)\}.$$

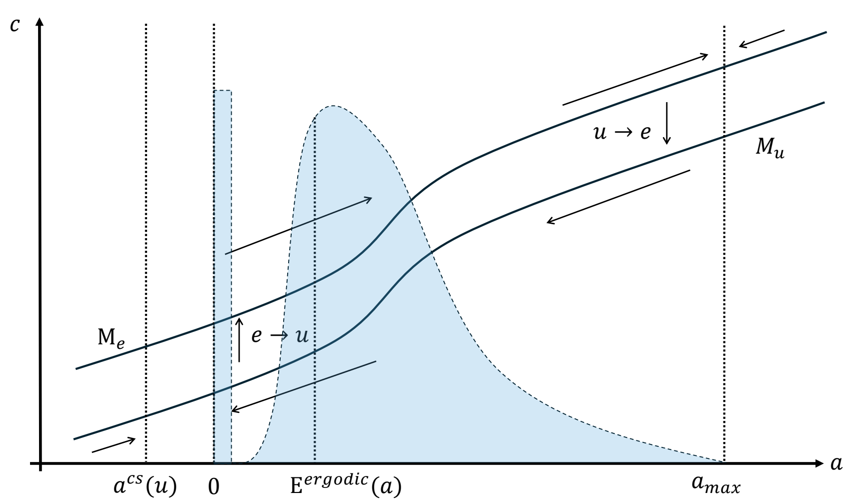
The individual conditional saddle under idiosyncratic state z *is the largest subset* $\mathcal{M}^{\text{ind}}_z(a_0, z_0) \subseteq \mathcal{R}^{\text{ind}}_z(a_0, z_0)$ *that is forward invariant under the frozen-*z *mapping:*

$$a \in \mathcal{M}^{\text{ind}}_z(a_0, z_0) \quad \Longrightarrow \quad g_a(a, z) \in \mathcal{M}^{\text{ind}}_z(a_0, z_0).$$

Equivalently,

$$\mathcal{M}^{\text{ind}}_z(a_0, z_0) := \bigcup \Big\{ \mathcal{S} \subseteq \mathcal{R}^{\text{ind}}_z(a_0, z_0) : \ g_a(\mathcal{S}, z) \subseteq \mathcal{S} \Big\}.$$

Figure B.1: Individual conditional saddle paths in the stationary RCE



Notes: The figure illustrates the individual conditional saddle paths for $z = e$ and $z = u$ in Aiyagari (1994).

Figure B.1 illustrates individual-level conditional saddle paths in Aiyagari (1994). Because of the borrowing constraint, the conditional steady state associated with the unemployment state $z = u$ is not attained. Under standard calibrations, this generates a positive mass of agents at the borrowing limit. Analogous to the aggregate-level case, heterogeneity in the slopes of individual conditional saddle paths implies differential responses of individual consumption to idiosyncratic shocks.

C Proofs for the theoretical results

Proposition 2 (Aggregate uncertainty and the conditional steady states)**.**
The following inequalities hold:

$$K_B^{cs} < K_B^{pf} < K_G^{pf} < K_G^{cs}, \qquad c_B^{cs} < c_B^{pf} < c_G^{pf} < c_G^{cs}.$$

Proof.
Let $R(A,K) := 1-\delta+\alpha A K^{\alpha-1}$ denote the gross return on capital and let the K-nullcline (feasibility locus) be

$$c(A,K) = AK^{\alpha} - \delta K.$$

The PF steady states solve the Euler equations under absorbing beliefs,

$$1 = \beta R(B, K_B^{pf}), \qquad 1 = \beta R(G, K_G^{pf}).$$

Since $R(A,K)$ is strictly increasing in A and strictly decreasing in K (because $\alpha - 1 < 0$), it follows immediately that $K_B^{pf} < K_G^{pf}$.

Next define the frozen-regime CS Euler residuals evaluated on the K-nullcline by

$$F_B(K) := \beta\Big[\pi_{BB}R(B,K) + \pi_{BG}\frac{c(B,K)}{c(G,K)}\,R(G,K)\Big] - 1,$$
$$F_G(K) := \beta\Big[\pi_{GG}R(G,K) + \pi_{GB}\frac{c(G,K)}{c(B,K)}\,R(B,K)\Big] - 1.$$

By construction, the conditional steady states satisfy $F_B(K_B^{cs}) = 0$ and $F_G(K_G^{cs}) = 0$.

Step 1: show $K_B^{cs} < K_B^{pf}$. Evaluate F_B at K_B^{pf}. Using $1 = \beta R(B, K_B^{pf})$ and $\pi_{BB} = 1 - \pi_{BG}$,

$$\begin{aligned} F_B(K_B^{pf}) &= \beta\Big[(1-\pi_{BG})R(B,K_B^{pf}) + \pi_{BG}\frac{c(B,K_B^{pf})}{c(G,K_B^{pf})}R(G,K_B^{pf})\Big] - 1 \\ &= \pi_{BG}\Big[\beta\frac{c(B,K_B^{pf})}{c(G,K_B^{pf})}R(G,K_B^{pf}) - 1\Big]. \end{aligned}$$

Thus $F_B(K_B^{pf}) < 0$ is equivalent to

$$\beta\frac{c(B,K_B^{pf})}{c(G,K_B^{pf})}R(G,K_B^{pf}) < 1 \quad\Longleftrightarrow\quad \frac{R(G,K_B^{pf})}{R(B,K_B^{pf})} < \frac{c(G,K_B^{pf})}{c(B,K_B^{pf})},$$

where we used $\beta R(B, K_B^{pf}) = 1$ to divide both sides by $R(B, K_B^{pf})$.

We now prove this strict inequality for any K with $c(B,K), c(G,K) > 0$. Write $x :=$

$\alpha K^{\alpha-1} > 0$ and $y := K^{\alpha} > 0$. Then

$$\frac{R(G,K)}{R(B,K)} = \frac{1-\delta+Gx}{1-\delta+Bx}, \qquad \frac{c(G,K)}{c(B,K)} = \frac{Gy-\delta K}{By-\delta K}.$$

Since $1-\delta > 0$, we have the strict bound

$$\frac{1-\delta+Gx}{1-\delta+Bx} < \frac{Gx}{Bx} = \frac{G}{B}.$$

Since $\delta K > 0$ and $Gy > By$, subtracting the same positive term from numerator and denominator enlarges the ratio, yielding

$$\frac{Gy-\delta K}{By-\delta K} > \frac{Gy}{By} = \frac{G}{B}.$$

Combining the two displays gives

$$\frac{R(G,K)}{R(B,K)} < \frac{G}{B} < \frac{c(G,K)}{c(B,K)},$$

and in particular the desired inequality holds at $K = K_B^{pf}$. Therefore $F_B(K_B^{pf}) < 0$.

Finally, note that F_B is strictly decreasing in K on the relevant region because both $R(B,K)$ and $R(G,K)$ are strictly decreasing in K and $c(B,K)/c(G,K)$ is also decreasing in K along the feasibility locus.[2] Hence, since $F_B(K_B^{cs}) = 0$ and $F_B(K_B^{pf}) < 0$, we must have $K_B^{cs} < K_B^{pf}$.

Step 2: show $K_G^{cs} > K_G^{pf}$. Similarly, evaluate F_G at K_G^{pf}. Using $1 = \beta R(G, K_G^{pf})$ and $\pi_{GG} = 1 - \pi_{GB}$,

$$\begin{aligned} F_G(K_G^{pf}) &= \beta\Big[(1-\pi_{GB})R(G,K_G^{pf}) + \pi_{GB}\frac{c(G,K_G^{pf})}{c(B,K_G^{pf})}R(B,K_G^{pf})\Big] - 1 \\ &= \pi_{GB}\Big[\beta\frac{c(G,K_G^{pf})}{c(B,K_G^{pf})}R(B,K_G^{pf}) - 1\Big]. \end{aligned}$$

Thus $F_G(K_G^{pf}) > 0$ is equivalent to

$$\beta\frac{c(G,K_G^{pf})}{c(B,K_G^{pf})}R(B,K_G^{pf}) > 1 \quad \iff \quad \frac{R(B,K_G^{pf})}{R(G,K_G^{pf})} > \frac{c(B,K_G^{pf})}{c(G,K_G^{pf})}.$$

But the argument above applied with (B,G) swapped gives, for any K with positive con-

[2] This monotonicity is standard and can be verified by differentiation; it is also visually apparent in the (K,C) phase diagram.

sumption,

$$\frac{R(B,K)}{R(G,K)} > \frac{B}{G} > \frac{c(B,K)}{c(G,K)}.$$

Hence $F_G(K_G^{pf}) > 0$. Since F_G is strictly decreasing in K and $F_G(K_G^{cs}) = 0$, we conclude $K_G^{cs} > K_G^{pf}$.

Step 3: conclude the ordering and translate to consumption. We have shown $K_B^{cs} < K_B^{pf}$ and $K_G^{pf} < K_G^{cs}$, and already $K_B^{pf} < K_G^{pf}$, hence

$$K_B^{cs} < K_B^{pf} < K_G^{pf} < K_G^{cs}.$$

Finally, along the K-nullcline $c(A,K) = AK^{\alpha} - \delta K$ is strictly increasing in A and (on the relevant region) increasing in K, so the same ordering carries over to consumption:

$$c_B^{cs} < c_B^{pf} < c_G^{pf} < c_G^{cs}.$$

■

References

Aiyagari, S. R. (1994). Uninsured Idiosyncratic Risk and Aggregate Saving. *The Quarterly Journal of Economics 109*(3), 659–684. Publisher: Oxford University Press.

Stokey, N. L., R. E. Lucas, and E. C. Prescott (1989). *Recursive Methods in Economic Dynamics.* Harvard University Press.