Appendix

Contents

A Propositions and proofs

Proposition 1 (The qualification for the sufficient statistic)**.**
Suppose a time series of the value (policy) functions $\{V_t\}_t^T$ *is given from a recursive competitive equilibrium that is unique. For a sufficiently large* T*, if there exists a time series of a variable* $\{e_t\}_{t=0}^T$ *such that for each time partition* $\mathcal{T}_S = \{t|S_t = S\}$*,* $\forall S \in \{B, G\}$ *and for* $\forall(a, z)$*,*

$$\begin{aligned}
&(i) \quad e_{\tau_0} < e_{\tau_1} \iff V_{\tau_0}(a, z) < V_{\tau_1}(a, z) \textit{ for any } \tau_0, \tau_1 \in \mathcal{T}_S \\
&or \\
&(ii) \quad e_{\tau_0} < e_{\tau_1} \iff V_{\tau_0}(a, z) > V_{\tau_1}(a, z) \textit{ for any } \tau_0, \tau_1 \in \mathcal{T}_S,
\end{aligned}$$

then e_t *is the sufficient statistic of the endogenous aggregate state* Φ_t *for* $\forall t$*.*

Proof.

It is sufficient to show that the following equivalence holds:

$$\{t \in \mathcal{T}_S | e_t = e_\tau\} = \{t \in \mathcal{T}_S | \Phi_t = \Phi_\tau\} \quad \text{for } \forall \tau \text{ and } \forall S \in \{B, G\}.$$

First, the following direction holds:

$$\{t \in \mathcal{T} | e_t = e_\tau\} \supseteq \{t \in \mathcal{T} | \Phi_t = \Phi_\tau\} \quad \text{for } \forall \tau \text{ and } \forall S \in \{B, G\}.$$

It is because if two periods share the same aggregate states (both endogenous and exogenous), the level of the time-specific value function is the same. This implies, $e_{\widetilde{t}} = e_\tau$. That is,

$$\text{For } \forall \widetilde{t} \in \{t \in \mathcal{T} | \Phi_t = \Phi_\tau\}, \quad V_{\widetilde{t}} = V_\tau \implies \text{For } \forall \widetilde{t} \in \{t \in \mathcal{T} | \Phi_t = \Phi_\tau\}, \quad e_{\widetilde{t}} = e_\tau.$$

Otherwise, the strict monotonicity condition (i) or (ii) is violated.

Second, we need to show

$$\{t \in \mathcal{T} | e_t = e_\tau\} \subseteq \{t \in \mathcal{T} | \Phi_t = \Phi_\tau\} \quad \text{for } \forall \tau \text{ and } \forall S \in \{B, G\}.$$

From the monotonicity condition, the following is true:

$$\text{For } \forall \widetilde{t} \in \{t \in \mathcal{T} | e_t = e_\tau\}, \quad V_{\widetilde{t}} = V_\tau \quad \text{for } \forall \tau \text{ and } \forall S \in \{B, G\}.$$

Then, it is sufficient to show that

$$V_{\widetilde{t}} = V_\tau \implies \Phi_{\widetilde{t}} = \Phi_\tau.$$

Suppose it is not true. Then, there exists $\widetilde{t}$ such that

$$V_{\widetilde{t}} = V_\tau \text{ and } \Phi_{\widetilde{t}} \neq \Phi_\tau.$$

This violates the primitive of the method, which is the uniqueness of the equilibrium, as all the agents become indifferent between the two different aggregate state realizations. That is, the multiple equilibrium path are possible, which contradicts the assumption.

Therefore, the following holds:

$$\text{For } \forall \widetilde{t} \in \{t \in \mathcal{T} | e_t = e_\tau\}, \quad \Phi_{\widetilde{t}} = \Phi_\tau.$$

This implies

$$\{t \in \mathcal{T} | e_t = e_\tau\} \subseteq \{t \in \mathcal{T} | \Phi_t = \Phi_\tau\}$$

for any τ and $S \in \{B, G\}$.

■

B The sufficient statistic validation

Figure B.1 plots the level of the policy functions in the vertical axis and the corresponding aggregate capital stock in the horizontal axis for an individual household with the median-level capital stock and the unemployed status (low idiosyncratic productivity) in the solution of Krusell and Smith (1998).[1] Each panel is for the different contemporaneous aggregate productivity levels. The monotonicity stays unaffected regardless of the choice of the individual household.

Figure B.1: Monotonicity of the policy functions in aggregate capital stock

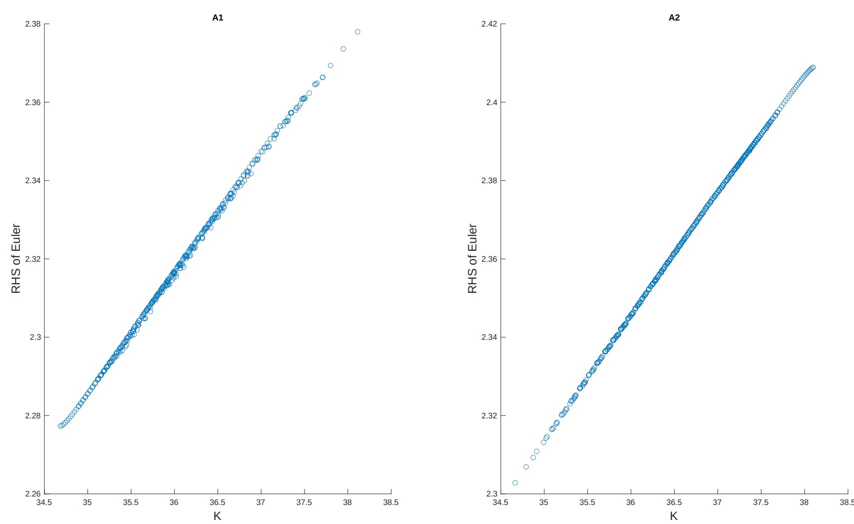
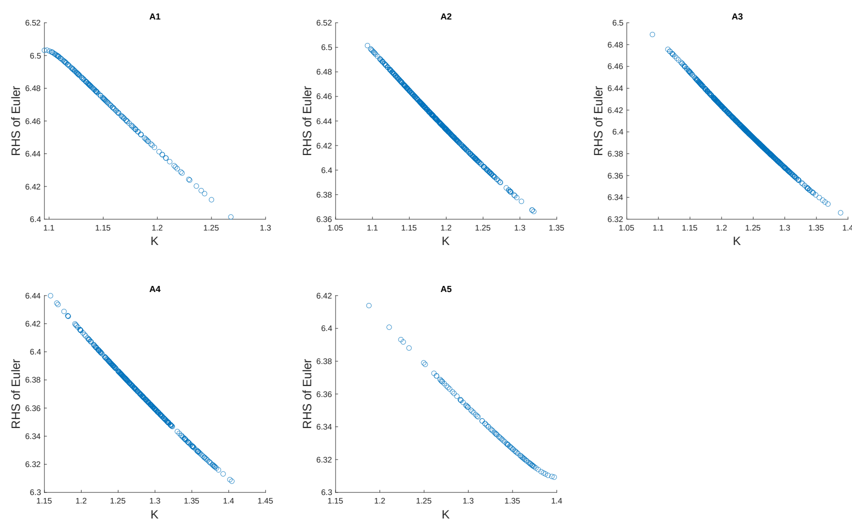


Figure B.2 plots the level of the value functions in the vertical axis and the the corresponding aggregate capital stock in the horizontal axis for an individual firm with the median-level capital stock and idiosyncratic productivity in the solution of Khan and Thomas (2008). Each panel is for the different contemporaneous aggregate productivity levels. The monotonicity stays unaffected regardless of the choice of the individual firm.

[1] It is worth noting that the plotted policy level is not the realized equilibrium allocation but a policy function.

Figure B.2: Monotonicity of the value functions in aggregate capital stock



C Parameter levels in the leading application

All the parameters are from Khan and Thomas (2008) except for the borrowing constraint parameter, which I used the level of Guerrieri and Iacoviello (2015).

Table C.1: Externally calibrated parameters

Parameter	Description	Value
α	capital share	0.256
γ	labor share	0.640
δ	depreciation	0.069
ϕ	borrowing constraint parameter	0.975
β	discount factor	0.977
η	labor disutility	2.400
ρ	idiosyncratic productivity persistence	0.859
σ	idiosyncratic productivity volatility	0.022
ρ_A	aggregate TFP persistence	0.859
σ_A	aggregate TFP volatility	0.014

References

GUERRIERI, L. AND M. IACOVIELLO (2015): "OccBin: A toolkit for solving dynamic models with occasionally binding constraints easily," *Journal of Monetary Economics*, 70, 22–38.

KHAN, A. AND J. K. THOMAS (2008): "Idiosyncratic Shocks and the Role of Nonconvexities in Plant and Aggregate Investment Dynamics," *Econometrica*, 76, 395–436.

KRUSELL, P. AND A. A. SMITH, JR. (1998): "Income and Wealth Heterogeneity in the Macroeconomy," *Journal of Political Economy*, 106, 867–896.