Computational Methods for Macroeconomics (Part II)
Lecture 4: Heterogeneous firm models

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Background: Heterogeneous firm models

- Prescott and Kydland (1982): Time-to-build for capital investment
- Hopenhayn (1992): stationary equilibrium distribution of firm-level allocations: entry, exit, and size.
- Capital adjustment cost
  - Abel (1983): Convex capital adjustment cost
  - Cooper and Haltiwanger (2006): "The Nature of Capital Adjustment Costs"
- Lumpy investment
  - Caballero and Bertola (1994), Caballero and Engel (1999), Abel and Eberly (2002)
  - Khan and Thomas (2008), Khan and Thomas (2003): Strong GE effect
  - Bachmann et al. (2013), Winberry (2021), Koby and Wolf (2020), Lee (2022): Weak GE effect
- Heavy tail distribution
  - Gabaix (2009): A heavy-tail of the firm size distribution
- Financial friction
  - Bernanke and Gertler (1989), Kiyotaki Moore (1997), Brunnermeier and Sannikov (2014)

Background: Khan and Thomas (2008)

- Khan and Thomas (2008) studies heterogeneous establishments (c.f., firms) under the aggregate productivity fluctuations.
- An improved investment-to-capital distribution compared to Khan and Thomas (2003).
- Establishment-level nonlinear investment dynamics: (S, s) cycle.
- Macro-level log-linear investment dynamics: strong general equilibrium effect.
- Basic ingredients:
  - Heterogeneous idiosyncratic productivity process under the incomplete market (time-to-build).
  - Aggregate TFP fluctuations (Krusell and Smith, 1997).
  - The fixed cost, $\xi \sim_{iid} Unif[0, \bar{\xi}]$: smoothing the kink of the value function.
  - A small-scale investment is allowed, which is not subject to a fixed cost.
  - Value function normalization steps.
  - Non-trivial market clearing condition.
  - Representative household and competitive factor market.

Model

Establishment-level production

- At the beginning of period t, a firm i is given with $(k_{it}, z_{it}; S_t)$:
  - k_{it}: Pre-determined establishment-level capital stock.
  - z_{it}: Establishment-level idiosyncratic productivity (AR(1) process).
  - $S_t = \{A_t, \Phi_t\}$: A_t is aggregate productivity (AR(1) process); Φ_t is the distribution of individual establishments.
- Cobb-Douglas production function with DRS ($\alpha + \gamma < 1$) where labor demand is contemporaneously determined:

$$f(k_{it}, z_{it}; S_t) = A_t z_{it} (k_{it})^{\alpha} n_{it}^{\gamma}$$

- Operating profit due to DRS: $\pi(k_{it}, z_{it}; S_t) = \max_{n_{it}} f(k_{it}, z_{it}; S_t) - w_t n_{it}$
- Operating profit = Dividends (D_{it}) + Investment (I_{it})
- The objective function is maximizing the firm value:

$$J_{it} = \max_{\{D_{it}\}_{t=0}^{\infty}} \sum_{t=0}^{\infty} \frac{1}{R_t} D_{it}$$

Establishment-level investment

- A firm needs to decide k_{it+1} by choosing I_{it}.

$$k_{it+1} = (1-\delta)k_{it} + I_{it}$$

- Two options in the investment scale: large/small
- If $I_{it} \in \Omega(k_{it}) := [\nu k_{it}, \nu k_{it}]$, then there is no fixed cost. $(\nu < \delta)$
- If $I_{it} \notin \Omega(k_{it}) := [\nu k_{it}, \nu k_{it}]$, then a fixed cost $\xi_{it} \sim_{iid} Unif([0, \overline{\xi}])$: Why?
- Role of fixed adjustment cost:
  - Inaction period: no lumpy investment - inside the (S, s) cycle.
  - Large adjustment: Jumping from s to S.
- No convex adjustment cost in Khan and Thomas (2008) but Winberry (2021) introduces a convex adjustment cost.
- Role of convex adjustment cost:
  - When your productivity jumps from 1 to 1.5, the unconstrained optimal level of future capital jumps identically across the different size of firms.
  - Without the convex adjustment cost, the capital stock of small firms can immediately jump up to the optimal level, being a sudden large firm.

Household

A representative household consumes, supplies labor, and saves.

$$V(a;S) = \max_{c,a',l_H} \quad log(c) - \eta l_H + \beta \mathbb{E} V(a';S')$$

$$\text{s.t. } c + \int \Gamma_{S,S'} q(S,S') a(S') dS' = w(S) l_H + \int a(S) dS$$

$$G_\phi(S) = \Phi', \quad G_A(A) = A', \quad S = \{\Phi, A\}$$

- a: an equity portfolio, Φ: distribution of firms
  A: aggregate productivity, c: consumption
  a': a state-contingent future saving portfolio, l_H: labor supply (indivisible)
  q: state-contingent bond price, w: wage
- Household is holding the equity of firms as their wealth.
- Stochastic discount factor:

$$q(S,S') = \beta \frac{C(S)}{C(S')}$$

Khan and Thomas (2008) defines $p(S) := \frac{1}{C(S)}$, which will be extensively used after the normalization.

Recursive formulation

$$J(k,z;S) = \pi(k,z;S) + (1-\delta)k + \int_0^{\overline{\xi}} \max\left\{R^*(k,z;S) - w(S)\xi, R^c(k,z;S)\right\} dG_\xi(\xi)$$

$$R^*(k,z;S) = \max_{k'} \; -k' - c(k,k') + \mathbb{E}m(S,S')J(k',z';S')$$

$$R^c(k,z;S) = \max_{k^c-(1-\delta)k\in\Omega(k)} \; -k^c - c(k,k^c) + \mathbb{E}m(S,S')J(k^c,z';S')$$

(Operating profit) $\pi(z,k;S) := \max_{n_d} zAk^\alpha n_d^\gamma - w(S)n_d$ (n_d: labor demand)

(Constrained investment) $I^c \in \Omega(k) := [-k\nu, k\nu] \quad (\nu < \delta)$

(Convex adjustment cost) $c(k,k') := \left(\mu^I/2\right)\left((k'-(1-\delta)k)/k\right)^2 k$

(Khan and Thomas (2008): $\mu = 0$)

(Idiosyncratic productivity) $z' = G_z(z)$ (AR(1) process)

(Stochastic discount factor) $m(S,S') = \beta\left(C(S)/C(S')\right)$

(Aggregate states) $S = \{A, \Phi\}$

(Aggregate law of motion) $\Phi' := H(S)$, $A' = G_A(A)$ (AR(1) process),

National accounting

- National account tracking is important for the efficient GE computation.

$$
\begin{aligned}
Y &= C + I = C + (\tilde{I} + Adj.Cost) \\
C &= Y - I \\
&= (\Pi + W * L) - I \\
&= (\Pi - I) + W * L \\
&= \underbrace{D}_{\text{Dividend income}} + \underbrace{W * L}_{\text{Labor income}}
\end{aligned}
$$

- Therefore, consumption is total dividends plus total labor expenses.
- After obtaining the distribution of firms, we compute total dividend and labor expense. Then, we obtain the consumption.
- Why does consumption matter? It determines $w(S)$ and $q(S, S')$: next slide.

Non-trivial market clearing condition

- From the intra-temporal labor supply optimality condition:

$$\begin{aligned}\eta &= \lambda(\mu(S); S)w(S) \\ &= p(S)w(S)\end{aligned}$$

- Therefore, if $p(S)$ is known, then $w(S)$ is determined. (What if GHH utility?)
- We still need to know SDF, $q(S, S')$, to solve the problem.
- However, the following slide's normalization eases the problem: $p(S)$ is the only price!
- Where is $p(S)$ is determined?
  - There is no closed-form to determine $P(S)$.
  - The notorious internal loop:
    1. Guess $p(S)$.
    2. Using the given distribution, $\Phi(S)$ compute the aggregate consumption $c(S) = D(S) + W(S) * L(S)$.
    3. Compute $p^{update}(S) = 1/c(S)$, and repeat the steps until $||p(S) - p^{update}(S)|| < tol$

Normalization

Multiply $p(S) = 1/C(S)$ on the both sides of the value function identity.

$$p(S)J(k,z;S) = p(S)(\pi(k,z;S) + (1-\delta)k) + \int_0^{\overline{\xi}} \max\left\{p(S)R^*(k,z;S) - p(S)w(S)\xi, p(S)R^c(k,z;S)\right\} dG_\xi(\xi)$$

Define $(\widetilde{J}, \widetilde{R}^*, \widetilde{R}^c)$ as follows:

$$\begin{aligned}
\widetilde{J}(k,z;S) &:= p(S)J(k,z;S) \\
\widetilde{R}^*(k,z;S) &:= p(S)R^*(k,z;S) && = \max_{k'} \; (-k' - c(k,k'))p(S) + \mathbb{E}p(S)q(S,S')J(k',z';S') \\
& && = \max_{k'} \; (-k' - c(k,k'))p(S) + \mathbb{E}{\color{red}\beta} p(S')J(k',z';S') \\
& && = \max_{k'} \; (-k' - c(k,k'))p(S) + \mathbb{E}{\color{red}\beta} \widetilde{J}(k',z';S') \\
\widetilde{R}^c(k,z;S) &:= p(S)R^c(k,z;S) && = \max_{k^c-(1-\delta k)\in\Omega(k)} \; (-k^c - c(k,k^c))p(S) + \mathbb{E}{\color{red}\beta} \widetilde{J}(k^c,z';S')
\end{aligned}$$

It is necessary to check whether the recursive form is preserved for the normalized value functions.

$$\widetilde{J}(k,z;S) = p(S)(\pi(k,z;S) + (1-\delta)k) + \int_0^{\overline{\xi}} \max\left\{\widetilde{R}^*(k,z;S) - P(S)w(S)\xi, \widetilde{R}^c(k,z;S)\right\} dG_\xi(\xi)$$

- Thanks to this normalization, we only need to track $p(S)$ instead of $q(S,S')$.

Smoothing the kink and the extensive margin

$$\int_0^{\bar{\xi}} \max\left\{R^*(k,z;S) - w(S)\xi, R^c(k,z;S)\right\} dG_\xi(\xi) \tag{1}$$

Then, there exists $\xi^*(k,z;S)$ such that

$$\begin{aligned} R^*(k,z;S) - w(S)\xi &> R^c(k,z;S) \quad \text{if } \xi < \xi^*(k,z;S) \\ R^*(k,z;S) - w(S)\xi &\leq R^c(k,z;S) \quad \text{if } \xi \geq \xi^*(k,z;S) \end{aligned}$$

Especially, $\xi^*(k,z;S) = \frac{R^*(k,z;S) - R^c(k,z;S)}{w(S)}$ is the closed-form characterzation.

- In the support of ξ, $[0,\xi^*)$ corresponds to large-scale investment and $[\xi^*,\bar{\xi}]$ corresponds to small-scale investment: define $\psi(k,z;S) := \frac{min\{\xi^*(k,z;S),\bar{\xi}\}}{\bar{\xi}}$
- With probability $\psi(k,z;S)$, a firm makes a large-scale investment.
- Eq (1) becomes a linear combination form: No Kink! (c.f., Discrete choice model)

$$\psi(k,z;S)\left(R^*(k,z;S) - w(S)\frac{\xi^*(k,z;S)}{2}\right) + (1-\psi(k,z;S))\left(R^c(k,z;S)\right).$$

Recursive competitive equilibrium

$(g_c, g_a, g_{l_H}, g_{k^*}, g_{k^c}, g_{\xi^*}, g_{n_d}, \widetilde{V}, \widetilde{J}, \widetilde{R}^*, \widetilde{R}^c, p, w)$ is a recursive competitive equilibrium if the following conditions are satisfied.

1. $g_c, g_{lH}, \widetilde{V}$ and g_a, solves the household's problem.
2. $g_{k^*}, g_{k^c}, g_{\xi^*}, g_{n_d}, \widetilde{J}, \widetilde{R}^*$, and $\widetilde{R}^c$ solve a firm's problem.
3. Market Clearing:

$$\text{(Labor Market)} \qquad g_{lH}(\Phi; S) = \int \left(g_{n_d}(k, z; S) + \left(\frac{g_{\xi^*}(k, z; S)}{\bar{\xi}} \right) \left(\frac{g_{\xi^*}(k, z; S)}{2} \right) k^{\zeta} \right) d\Phi$$

$$\begin{aligned} \text{(Product Market)} \qquad g_c(\Phi; S) = \int \Bigg(& zAk^{\alpha} g_{n_d}(k, z; S)^{\gamma} \\ & - \Big((g_{k^*}(k, z; S) - (1-\delta)k) + c(k, g_{k^*}(k, z; S)) \Big) \frac{g_{\xi^*}(k, z; S)}{\bar{\xi}} \\ & - \Big((g_{k^c}(k, z; S) - (1-\delta)k) + c(k, g_{k^c}(k, z; S)) \Big) \frac{1 - g_{\xi^*}(k, z; S)}{\bar{\xi}} \Bigg) d\Phi \end{aligned}$$

4. Consistency Condition: ▸ Detail

Computation

Roadmap

- The following is the roadmap for computation section.
  1. Set the parametric law of motion: assumption
  2. Guess the parameters: $\#(S) \times 2 \times 2$
  3. Solution (optimization)
     - VFI/PFI/EGM/Projection method
     - Interpolation
  4. Simulation and internal loop for price p
     - Simulation
     - Aggregation
     - Update p until convergence
  5. Update the parameters
  6. After convergence, verify the assumption
- After this, we will talk about more recent developments.

Basic setup

The basic steps.

- Set directories
- Set parameters (might be a function argument)

Then, two important steps follow.

- Setting grid points.
  - Individual capital grid
  - Aggregate capital grid: this grid can be sparse (5∼10 grids)
- Discretizing idiosyncratic shock process (Markov chain)
  - Tauchen method
  - Rouwenhorst method

Parametric law of motion

There are two layers of choices:

- First, we need to set what are **sufficient statistics** to characterize the dynamics of the individual state distribution.
  - A good candidate is the first moment of the endogenous individual state.
  - As in Krusell and Smith (1998), by tracking only K_t, the aggregate prices are also characterized (Median also works well).
- Then, we need to decide the **parametric form** of the law of motion.
- So start from the following parameter guesses $(\alpha_S^K, \beta_S^K; \alpha_S^p, \beta_S^p)$

$$
\begin{aligned}
log(K_{t+1}) &= \alpha_S^K + \beta_S^K log(K_t) \quad \text{when } S_t = S \\
log(p_t) &= \alpha_S^p + \beta_S^p log(K_t) \quad \text{when } S_t = S
\end{aligned}
$$

- K_t does not immediately give p_t (no closed-form). But it should give some inference on p_t!

Computation - GE: Solution

Value function iteration with accelerator

Now we know $p(S)$, if we are given with K. The pseudo code is as follows:

1. Guess $J^{(n)} : \mathcal{K} \times \mathcal{Z} \times \mathcal{A} \times \mathcal{K}^{Agg} \to \mathbb{R}$
2. Solve for the policy function, $g_k^{(n)}$ (using monotonicity: $g_k^{(n)} \geq g_{\widetilde{k}}^{(n)}$ for $k \geq \widetilde{k}$).
   - We have K, so we know (p, K') from the law of motion.
   - Interpolate the value functions over K' to have $J^{(n)}(\cdot, z'; A', K')$
   - Then, the problem becomes a typical VFI.
3. Update $J^{(n+1)}$ using the policy function, $g_k^{(n)}$.
4. Update $J^{(n+2)}$ using the policy function, $g_k^{(n)}$.

   ...

5. Update $J^{(n+m)}$ using the policy function, $g_k^{(n)}$.
6. Check if $||J^{(n+m)} - J^{(n+m-1)}||_p < Tol$
   - If yes, the solution converged.
   - If no, go back to step 1.

Computation - GE: Simulation

Non-trivial market clearing condition (Revisited)

We use a non-stochastic iteration method. (Eigenvalue method is not feasible).

- Simulate a long enough aggregate shock path using Γ_S. (Not idiosyncratic shocks)
- Start from an initial guess $\mathcal{H}_0$. Compute the corresponding K_0. And then internal loop.
  - Guess p_0, and solve the problem to get g_{d0} and g_{l0}.
  - Compute c_0 using $(\mathcal{H}_0, g_{d0}, g_{l0}, w_0)$.
  - Compute $p_0^{update} = 1/c_0$, and repeat the steps until $||p_0 - p_0^{update}|| < tol$.
  - So, we have $(K_0, p_0^{converged})$.
- Let $\mathcal{H}_0$ evolve to $\mathcal{H}_1$ using $g_{a0}^{converged}$, and compute the corresponding K_1.
  - Guess p_1, and solve the problem to get g_{d1} and g_{l1}.
  - Compute c_1 using $(\mathcal{H}_1, g_{d1}, g_{l1}, w_1)$.
  - Compute $p_1^{update} = 1/c_1$, and repeat the steps until $||p_1 - p_1^{update}|| < tol$.
  - So, we have $(K_1, p_1^{converged})$.
- ...
- By repeating this process, we obtain $\{K_t, p_t^{converged}\}_{t=0}^{T}$.
- Discard the the burn-in period to get $\{K_t, p_t^{converged}\}_{t=burnIn}^{T}$.

Updating the parameters

- We have $\{K_t, p_t^{converged}\}_{t=burnIn}^{T}$ and $\{S_t\}_{t=burnIn}^{T}$.
- Fit the time-series into the parametric form of the law of motion to estimate the parameters: $(\alpha_S^K, \beta_S^K; \alpha_S^p, \beta_S^p)$

$$log(K_{t+1}) = \alpha_S^K + \beta_S^K log(K_t) \quad \text{when } S_t = S$$
$$log(p_t^{converged}) = \alpha_S^p + \beta_S^p log(K_t) \quad \text{when } S_t = S$$

- If the parameter estimates are not close to the guess, return to the initial step.
- Otherwise, the solution is converged.
- Check R^2 as the first check for the validity of the parametric form.

Close-to-perfect aggregation

TABLE A.II

FORECASTING RULES IN FULL LUMPY MODEL

Productivity[a]	β_0	β_1	S.E.	Adj. R^2
A. Forecasting m_1'				
z_1 (119 obs)	0.009	0.800	0.15e−3	1.0000
z_2 (298 obs)	0.016	0.798	0.22e−3	0.9999
z_3 (734 obs)	0.023	0.796	0.23e−3	0.9999
z_4 (1,208 obs)	0.030	0.795	0.26e−3	0.9999
z_5 (1,682 obs)	0.037	0.794	0.27e−3	0.9999
z_6 (1,871 obs)	0.044	0.079	0.28e−3	0.9999
z_7 (1,706 obs)	0.051	0.793	0.26e−3	0.9999
z_8 (1,237 obs)	0.058	0.792	0.24e−3	0.9999
z_9 (751 obs)	0.065	0.792	0.23e−3	0.9999
z_{10} (295 obs)	0.072	0.791	0.25e−3	0.9999
z_{11} (99 obs)	0.079	0.791	0.19e−3	0.9999
B. Forecasting p				
z_1 (119 obs)	0.994	−0.397	0.03e−3	1.0000
z_2 (298 obs)	0.986	−0.395	0.04e−3	1.0000
z_3 (734 obs)	0.977	−0.394	0.04e−3	1.0000
z_4 (1,208 obs)	0.968	−0.393	0.05e−3	1.0000
z_5 (1,682 obs)	0.958	−0.392	0.05e−3	1.0000
z_6 (1,871 obs)	0.949	−0.391	0.05e−3	1.0000
z_7 (1,706 obs)	0.940	−0.389	0.05e−3	1.0000
z_8 (1,237 obs)	0.931	−0.388	0.05e−3	1.0000
z_9 (751 obs)	0.921	−0.386	0.04e−3	1.0000
z_{10} (295 obs)	0.912	−0.384	0.05e−3	1.0000
z_{11} (99 obs)	0.903	−0.382	0.04e−3	1.0000

[a] Forecasting rules are conditional on current aggregate total factor productivity z_i. Each regression takes the form $\log(y) = \beta_0 + \beta_1 \log(m_1)$, where $y = m_1'$ or p.

Notes: The table is from Khan and Thomas (2008).

Concluding remarks

Summary

- Khan and Thomas (2008) provides a great benchmark to start with.
  - The micro-level non-linearity washes out if the general equilibrium effect is strong enough.
  - Strength of the general equilibrium effect depends on the price-elasticities of agents.
  - Are the price-elasticities at an empirically-supported range?
- The normalization technique is a great idea: See Winberry (2021) and Lee (2022).
- Thanks to the close-to-perfect aggregation, the algorithm of Krusell and Smith (1997) perfectly works.
  - The log-linear law of motion of sufficient statistics perfectly governs the aggregate dynamics.
  - No closed-form for $p(S)$: the internal loop is needed at a computational cost.
- What about financial frictions?
  - Consult with Ferreira, Haber, and Rörig (2021).
  - Real friction vs. Financial friction

Recursive competitive equilibrium: consistency condition ▸ Back

$$(\text{Consistency}) \quad G_\Phi(\Phi) = H(\Phi) = \Phi', \text{ where for } \forall K' \subseteq \mathbb{K} \text{ and } z' \in \mathbb{Z},$$

$$\Phi'(K', z') = \int \Gamma_{z,z'} \Bigg(\mathbb{I}\{g_{k^*}(k, z; S) \in K'\} \frac{g_{\xi^*}(k, z; S)}{\bar{\xi}} + \mathbb{I}\{g_{k^c}(k, z; S) \in K'\} \frac{1 - g_{\xi^*}(k, z; S)}{\bar{\xi}} \Bigg) d\Phi$$