Computational Methods for Macroeconomics (Part II)
Lecture 3: A heterogeneous household model in continuous time

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Background: Heterogeneous agent model in continuous time

- The long history of continuous-time modeling in finance.
  - Partial differential equation from Physics.
  - Ito's lemma.
  - e.g., Black-Scholes option pricing model.
  - Utility-free arguments based on no-arbitrage conditions (option pricing is doable with the given price).
- Recent development in continuous-time modeling in macro:
  - Macro with the financial sector: Brunnermeier and Sannikov (2014).
  - Search and match (OTC market): Duffie, Garleanu, and Pedersen (2005).
  - Incomplete market: Moll et al. (2021)
- Pros:
  - Tractability: the dynamics is characterized by a PDE.
  - Independence from shock orders: everything happens instantaneously with a probability.
  - Computational efficiency (next slide).
- Cons:
  - The unit of time is unmatched with the data side.

Background: Moll et al. (2021)

- In the end, very similar to the discrete model approach:
  - Solution = Hamilton-Jacobi-Bellman equation (HJB).
  - Simulation = Kolmogorov Forward equation (KF).
- Computational gain:
  - Side-stepping borrowing constraint.
  - Tomorrow = Today, thus FOC becomes static.
  - Sparse linear system: computation is faster with numbers only around the diagonal.
  - Solution (HJB) immediately gives the distribution (KF).
    - This is similar to the non-stochastic eigenvector method.
    - But in continuous time, it becomes more immediate.
- We will use Finite-Difference (FD) method, which converges to the viscosity solution under certain conditions. (Unfortunately, no closed form in the full model.)
- Sparse matrix: computationally very easy to handle.
- A restrictive assumption can lead to a closed-form solution.

Discrete to Continuous

Continuous modeling 101 - Sequantial formulation

- Unit period length: Δ
- All flows are now expressed in terms of rate. (e.g., temporal utility $= u(c_t)\Delta$)
- Control variables: saving, s_t and consumption, c_t

$$v(a) = \max_{\{c_t\}_{t=0}^{\infty}} \sum_{t=0}^{\infty} \left(\frac{1}{1+\rho\Delta}\right)^{\frac{t}{\Delta}} u(c_t)\Delta$$

Take $\Delta \to 0$. Then,

$$\lim_{\Delta\to 0} \left(\frac{1}{1+\rho\Delta}\right)^{\frac{t}{\Delta}} = \lim_{\Delta\to 0} \left((1+\rho\Delta)^{\frac{1}{\Delta}}\right)^{-t} = \left(\lim_{\Delta\to 0} (1+\rho\Delta)^{\frac{1}{\Delta}}\right)^{-t} = e^{-\rho t}$$

As $\sum_{t=0}^{\infty} \Delta$ becomes a Rieman integration, the following equation holds:

$$v(a) = \lim_{\Delta\to 0} \max_{\{c_t\}_{t=0}^{\infty}} \sum_{t=0}^{\infty} \left(\frac{1}{1+\rho\Delta}\right)^{\frac{t}{\Delta}} u(c_t)\Delta = \max_{\{c_t\}} \int_0^{\infty} e^{-\rho t} u(c_t)dt$$

Continuous modeling 102 - Recursive formulation

- Unit period length: Δ
- All flows are now expressed in terms of rate. (e.g., temporal utility $= u(c_t)\Delta$)
- State variable: wealth, a_t
- Control variables: saving, s_t and consumption, c_t

$$v_t(a_t) = \max_{c_t, s_t} u(c_t)\Delta + \frac{1}{1+\rho\Delta} v_{t+\Delta}(a_t + s_t)$$
$$\text{s.t.} \quad c_t\Delta + s_t = a_t r_t \Delta$$

- Denote the optimal choices as $\widehat{s}_t$ and $\widehat{c}_t$
- Divide the both sides of the budget constraint by Δ:

$$c_t + \frac{s_t}{\Delta} = a_t r_t$$

- Define the optimal average saving rate $g_t := \frac{\widehat{s}_t}{\Delta}$.

Average rate of value change

Using the optimal allocations, we can rewrite the bellman equation:

$$v_t(a_t) = u(\widehat{c}_t)\Delta + \frac{1}{1+\rho\Delta} v_{t+\Delta}(a_t + g_t\Delta)$$

Move the left-hand side term to the right.

$$0 = u(\widehat{c}_t)\Delta + \underbrace{\overbrace{\frac{1}{1+\rho\Delta} v_{t+\Delta}(a_t + g_t\Delta)}^{\text{future value}} - \overbrace{v_t(a_t)}^{\text{original value}}}_{\text{total value variation in p.v. over time } \Delta}$$

Divide both sides by $\Delta > 0$.

$$0 = \underbrace{u(\widehat{c}_t)}_{\text{Instantneous utility rate}} + \underbrace{\frac{1}{\Delta}\left(\frac{1}{1+\rho\Delta} v_{t+\Delta}(a_t + g_t\Delta) - v_t(a_t)\right)}_{\text{Average rate of value change}}$$

Instantneous rate of value change: $\Delta \to 0$

Take $\Delta \to 0$.

$$0 = \underbrace{u(\widehat{c}_t)}_{\text{Instantneous utility rate}} + \underbrace{\lim_{\Delta\to 0}\frac{1}{\Delta}\left(\frac{1}{1+\rho\Delta}v_{t+\Delta}(a_t+g_t\Delta) - v_t(a_t)\right)}_{\text{Instantneous rate of value change}}$$

Define $h(\Delta) := \frac{1}{1+\rho\Delta}v_{t+\Delta}(a_t+g_t\Delta)$. Note that $h(0) = v_t(a_t)$.

$$\begin{aligned} 0 &= \underbrace{u(\widehat{c}_t)}_{\text{Instantneous utility rate}} + \underbrace{\lim_{\Delta\to 0}\frac{h(\Delta)-h(0)}{\Delta - 0}}_{\text{Instantneous rate of value change}} \\ &= \underbrace{u(\widehat{c}_t)}_{\text{Instantneous utility rate}} + \underbrace{h'(\Delta)|_{\Delta=0}}_{\text{Instantneous rate of value change}} \end{aligned}$$

Instantneous rate of value change: $\Delta \to 0$ (cont'd)

From the chain rule,

$$\begin{aligned} h'(\Delta) = &-\frac{\rho}{(1+\rho\Delta)^2} v_{t+\Delta}(a_t + g_t\Delta) \\ &+ \frac{1}{1+\rho\Delta}\frac{\partial}{\partial\Delta}\left(v_{t+\Delta}(a_t + g_t\Delta)\right) \\ &+ \frac{1}{1+\rho\Delta}\frac{\partial}{\partial a_t}\left(v_{t+\Delta}(a_t + g_t\Delta)\right) g_t \end{aligned}$$

We project $h'(\Delta)$ onto $\Delta = 0$:

$$h'(\Delta)|_{\Delta=0} = -\rho v_t(a_t) + \dot{v}_t(a_t) + v_t'(a_t) g_t$$

where $\dot{v}_t(a_t) = \frac{\partial v_t(a_t)}{\partial t}$.

Hamilton-Jacobi-Bellman (HJB) equation

Therefore,

$$\begin{aligned} 0 &= u(\widehat{c}_t) - \rho v_t(a_t) + \dot{v}_t(a_t) + v_t'(a_t) g_t \\ &= u(\widehat{c}_t) - \rho v_t(a_t) + \dot{v}_t(a_t) + v_t'(a_t)(a_t r_t - \widehat{c}_t). \end{aligned}$$

In the conventional form,

$$\underbrace{\rho v_t(a_t)}_{\text{Instantaneous value rate}} = u(\widehat{c}_t) + \dot{v}_t(a_t) + v_t'(a_t)(a_t r_t - \widehat{c}_t).$$

which is a PDE.

- What's PDE?
  - An equation that relates partial derivatives of unknown function (v) with respect to independent variables (a, t): a is spatial and t is time.
  - Infinite number of solutions to PDE without boundary + initial conditions:
    - Boundary conditions: $v_t(a) = \phi_t(a)$ (Sum of orders of highest partial derivatives in each spatial variable $= 1$)
    - Initial condtiion(s): $v_0(a) = v_0$ (Highest order of time derivative $= 1$)

Basic problem

With the max operator, everything follows smoothly. Now, we can formulate the problem as follows:

$$\rho v_t(a) = \max_{c_t} u(c_t) + v_t'(a_t)(a_t r_t - c_t) + \dot{v}_t(a_t)$$

In a stationary environment, what we impose is

$$v_t(a) = v(a) \quad \text{for } \forall (a,t) \text{ and } \lim_{T\to\infty} e^{-\rho T} v(a) = 0 \text{ for } \forall a \text{ (TVC)}.$$

These are equivalent to the initial and boundary conditions. Then, we have

$$\rho v(a) = \max_{c} u(c) + v'(a)(ar - c)$$

Time does not matter anymore like a static problem: This is what Moll et al. (2021) calls as, "Today is tomorrow."

- Under the stationarity, the problem reduces down to ODE!
- This problem with CRRA utility has a closed-form solution.
  - e.g., log-utility: $c(a) = \rho a$ and $v(a) = \frac{1}{\rho} a + \textit{constant}$

Model

Revisiting Aiyagari (1994)

- We consider the incomplete market economy as in Aiyagari (1994).
- The economy stays in the stationary environment.
- A borrowing constraint: $a_t \geq 0$.
- For now, let's assume a Poisson endowment process $z_1 < z_2$ where a jump from state i to the other happens at the intensity of λ_i.

$$
\begin{aligned}
v_i(a) = \max_{c,s} u(c)\Delta + \frac{1}{1+\rho\Delta}((1-\lambda_i)v_i(a+s) + \lambda_i v_j(a+s)) \\
\text{s.t.} \quad c\Delta + s = wz_i\Delta + ar\Delta \\
a + s \geq 0
\end{aligned}
$$

- Note that there is no time index as the stationarity is already applied.
- Consider the following rearrangement:

$$
v_i(a) = \max_{c,s} u(c)\Delta + \frac{1}{1+\rho\Delta}(v_i(a+s) + \lambda_i(v_j(a+s) - v_i(a+s)))
$$

Hamilton-Jacobi-Bellman equation

Hamilton-Jacobi-Bellman equation (HJB)

$$\rho v_i(a) = \max_c u(c) + v_i'(a)(wz_i + ar - c) + \lambda_i(v_j(a) - v_i(a))$$

where $\quad a \in [0, \infty)$

- The poisson process can be generalized to Ornstein-Uhlenbeck process which is a continuous-time counterpart of AR(1) process (Drift + Brownian terms).
- We will solve this equation using FD.
- Now, we have a state boundary condition $a \in [0, \infty)$.
- FOC gives

$$u'(c_i(a)) = v_i'(a)$$

- At the boundary $a = 0$, optimal saving needs to be non-negative: $wz_i - c \geq 0 \implies wz_i \geq c$.
- Thus, boundary condition for ODE: $v_i'(0) = u_i'(c(0)) \geq u'(wz_i)$.
- The occasionary binding constraint shows up only as a boundary condition: the side-stepping.

Kolmogorov Forward equation

Kolmogorov Forward equation (KF)

- Also known as Fokker-Planck equation.
- Stationary probability density $g_i(a)$ solves the following ODE:

$$\frac{d}{da}(s_i(a)g_i(a)) = -\lambda_i g_i(a) + \lambda_j g_j(a)$$

where $s_i(a) = (wz_i + ar - c_i(a))$.

- It is from stationary cumulative distribution $G_i(a)$ that solves

$$\dot{G}_i(a) = 0 = -\underbrace{s_i(a)g_i(a)}_{\text{Variation in the border}} - \underbrace{\lambda_i G_i(a)}_{\text{Jump outflow}} + \underbrace{\lambda_j G_j(a)}_{\text{Jump inflow}}$$

- Note that $s_i > 0$ implies moving out of $G_i(a)$.
- Formal derivation: Moll et al. (2021) online supplementary data B.3.
- From the solution s_i we obtained from FD, we will solve g_i.
- But, surprisingly, by solving s_i, g_i is already given! (In computation section)

Market clearing condition

Market clearing condition

- The aggregate capital K can be obtained from the following equation:

$$K = \sum_{i=i,2} \int_0^\infty a g_i(a) da$$

$$r(\Phi) = MPK(\Phi) - \delta = \alpha \left(\frac{K(\Phi)}{L} \right)^{\alpha - 1} - \delta$$

$$w(\Phi) = MPL(\Phi) = (1 - \alpha) \left(\frac{K(\Phi)}{L} \right)^{\alpha}$$

where $L = \sum_{i=1,2} z_i \int_0^\infty g_i(a) da$ is exogenously given. Why?

- Poisson rate gives the stationary marginal distribution of labor productivity.
- Let x be the stationary mass of state 1.
- $x\lambda_1 = (1 - x)\lambda_2$ should hold under the stationarity.
- $x = \frac{\lambda_2}{\lambda_1 + \lambda_2} = \int_0^\infty g_1(a) da$.

Computation

Roadmap

- Computation steps are even simpler than in discrete time models.
- We will compute SRCE where we rely on tracking K_t.
- The following is the roadmap for the computation section.
  1. Price (aggregate allocation) guess
  2. Solution (optimization)
     - Finite difference method for HJB equation
     - Viscosity solution
  3. Simulation
     - The transpose problem (adjoint relationship)
  4. Aggregation
  5. Price update

Basic setup

- Set directories
- Set parameters (might be a function argument)

Then, two important steps follow.

- Setting grid points.
  - Moll et al. (2021) uses equi-spaced grids but finer grids for smaller wealth also work.
- Setting idiosyncratic labor productivity process:
  - Poisson densities $\{\lambda_1, \lambda_2\}$ need to be determined.
  - Labor productivities $\{z_1, z_2\}$ need to be determined.
- Define size of value funtion: $(2 * \#wealthgrid) \times 1$.
- Define size of stationary distribution: $(2 * \#wealthgrid) \times 1$.
  - Both are vectorized!

Price guess

- We need to come up with a price (an aggregate allocation) as an initial guess.
- Consistent with the previous lectures, let's start with K.

Computation - GE: Solution

Computation strategy

Using the first-order condition, the following is the summary of the problem:

$$\rho v_i(a) = u(u'^{-1}(v_i'(a))) + v_i'(a)(s_i(a)) + \lambda_i(v_j(a) - v_i(a))$$
$$0 = -\frac{d}{da}(s_i(a)g_i(a)) - \lambda_i g_i(a) + \lambda_j g_j(a)$$

where $s_i = wz_i + ar - u'^{-1}(v_i'(a))$. We translate this problem into the following problem:

$$\rho v = u(v) + A(v;K)v$$
$$0 = A(v;K)^T g$$

How? By a clever "discretization". So this is not a theoretical outcome.

- Potentially applicable to other problems!

Finite difference (FD) method

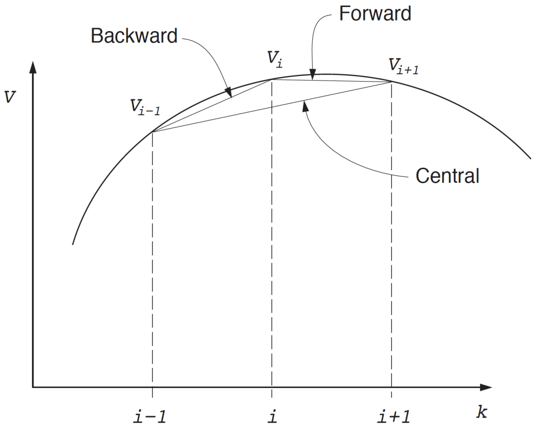
We will approximate v_i' using a finite difference between two grid points. Denote Δ as the distance between the points.

$$v_i'(a) \approxeq \frac{v_i(a) - v_i(a-\Delta)}{\Delta} \qquad \text{(Backward)}$$

$$v_i'(a) \approxeq \frac{v_i(a+\Delta) - v_i(a)}{\Delta} \qquad \text{(Forward)}$$

$$v_i'(a) \approxeq \frac{v_i(a+\Delta) - v_i(a-\Delta)}{2\Delta} \qquad \text{(Central difference)}$$

Different FD methods



Notes: The figure is from Benjamin Moll's lecture note.

Upwind scheme

We take

- forward difference whenever drift of state variable, s is $(+)$
- backward difference whenever drift of state variable, s is $(-)$

That is,

$$\rho v_i(a) = u(u'^{-1}(v_i'(a))) + \frac{v_i(a+\Delta) - v_i(a)}{\Delta} s_i^+(a) + \frac{v_i(a) - v_i(a-\Delta)}{\Delta} s_i^-(a) + \lambda_i(v_j(a) - v_i(a)). \tag{1}$$

where $s_i^+ = \max\{s_i, 0\}$ and $s_i^- = \min\{s_i, 0\}$

- This scheme is called "upwind scheme."
- Why? 1) Monotonicity condition 2) Boundary conditions from both sides.
- Solution for (1) is not trivial, but we can solve it like a discrete problem. (In two slides)
- The solution converges to the unique viscosity solution (end of the lecture).

Sparse matrix

With a slight abuse of the notation at this stage,

$$\begin{aligned}\rho v_i(a) = {} & u(u'^{-1}(v_i'(a))) + \frac{v_i(a+\Delta) - v_i(a)}{\Delta} s_i^+(a) + \frac{v_i(a) - v_i(a-\Delta)}{\Delta} s_i^-(a) \\ & + \lambda_i (v_j(a) - v_i(a)). \\ & = f(v(a)) + A(v(a); K) v(a)\end{aligned}$$

where

$$A(v(a); K) v(a) = \left[0 \cdots, \frac{s_i^+(a)}{\Delta}, \frac{-s_i^+(a) + s_i^-(a)}{\Delta} - \lambda_i, \frac{-s_i^-(a)}{\Delta}, \cdots, \lambda_i, \cdots, 0 \right] \begin{bmatrix} \cdots \\ v_i(a+\Delta) \\ v_i(a) \\ v_i(a-\Delta) \\ \cdots \end{bmatrix}$$

Sparse matrix (cont'd)

$$
\boldsymbol{A}^n = \begin{bmatrix}
y^n_{1,1} & z^n_{1,1} & 0 & \cdots & \lambda_1 & 0 & 0 & \cdots & 0 \\
x^n_{1,2} & y^n_{1,2} & z^n_{1,2} & \cdots & 0 & \lambda_1 & 0 & \cdots & 0 \\
0 & x^n_{1,3} & y^n_{1,3} & z^n_{1,3} & \cdots & 0 & \lambda_1 & \cdots & 0 \\
\vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \vdots \\
0 & \ddots & x^n_{1,J} & y^n_{1,J} & 0 & 0 & 0 & 0 & \lambda_1 \\
\lambda_2 & 0 & 0 & \cdots & y^n_{2,1} & z^n_{2,1} & 0 & \cdots & 0 \\
0 & \lambda_2 & 0 & \cdots & x^n_{2,2} & y^n_{2,2} & z^n_{2,2} & 0 & \cdots \\
0 & 0 & \lambda_2 & \cdots & 0 & x^n_{2,3} & y^n_{2,3} & z^n_{2,3} & 0 \\
\vdots & \ddots & \ddots & \vdots & \vdots & \ddots & \ddots & \ddots & \ddots \\
0 & 0 & 0 & \lambda_2 & 0 & \cdots & 0 & x^n_{2,J} & y^n_{2,J}
\end{bmatrix}
$$

Notes: The figure is from Jesús Fernández-Villaverde's lecture note.

Solving the nonlinear problem I

The solution method utilizes the stationary condition $\dot{v}(a) = 0$.

$$
\begin{aligned}
\frac{v^{(n+1)}(a) - v^{(n)}(a)}{dt} + \rho v_i^{(n)}(a) = {} & u(u'^{-1}(v_i^{(n)'}(a))) \\
& + \frac{v_i^{(n)}(a+\Delta) - v_i^{(n)}(a)}{\Delta} s_i^+(a) + \frac{v_i^{(n)}(a) - v_i^{(n)}(a-\Delta)}{\Delta} s_i^-(a) \\
& + \lambda_i (v_j^{(n)}(a) - v_i^{(n)}(a))
\end{aligned}
$$

where $s_i(a) = (wz_i + ar - u'^{-1}(v_i^{(n)'}(a)))$, and $dt > 0$ is any small number.

1. Guess $v^{(n)}$
2. Compute all except for $v^{(n+1)}$.
3. Update $v^{(n+1)}$ until the convergence.

This method is called as an "explicit" method by Moll et al. (2021).
The "Implicit" method is faster thanks to the sparsity of A.

Solving the nonlinear problem II

Now assume the following holds:

$$
\begin{aligned}
\frac{v^{(n+1)}(a) - v^{(n)}(a)}{dt} &+ \rho v_i^{(n)}(a) = u(u'^{-1}(v_i^{(n)'}(a))) \\
&+ \frac{v^{(n+1)}{}_i(a+\Delta) - v^{(n+1)}{}_i(a)}{\Delta} s_i^{+}(a) + \frac{v^{(n+1)}{}_i(a) - v^{(n+1)}{}_i(a-\Delta)}{\Delta} s_i^{-}(a) \\
&+ \lambda_i(v_j^{(n)}(a) - v_i^{(n)}(a)) \\
&= u(u'^{-1}(v_i^{(n)'}(a))) + A(v^{(n)}) v^{(n+1)} \\
\implies \left(\rho + \frac{1}{dt} - A(v^{(n)})\right) v^{(n+1)}(a) &= u(u'^{-1}(v_i^{(n)'}(a))) + \frac{v^{(n)}(a)}{dt}
\end{aligned}
$$

The sparcity of $A(v^{(n)})$ gives a speed boost.

Transpose problem

Now we have the solution (v, c, s, A). Then, we need to let the distribution evolve following KF:

$$0 = -\frac{d}{da}(s_i(a)g_i(a)) - \lambda_i g_i(a) + \lambda_j g_j(a)$$

Using the expansion of the derivative using FD, we can verify that the following equation holds:

$$0 = A(v; K)^T g$$

- Finding g is an eigenvector problem: the same approach as the discrete-time histogram method using eigenvector.
- Normalize g to satisfy $\int_0^\infty g_1(a)da + \int_0^\infty g_2(a)da = 1$.

Aggregation

- Compute the aggregate allocations using the stationary distribution g_i.

$$K^{update} = \sum_{i=i,2} \int_0^{\infty} a g_i(a) da$$

- Update the guess, K^{guess}, until the convergence between the guess and the implied level, K^{update}.

The following slides are helpful for understanding Viscosity solution.
These are from the lecture slides of Jesús Fernández-Villaverde.

Why does the method work?

- Well-developed theory for numerical solution of HJB equation using finite difference methods.

- Barles and Souganidis (1991), "Convergence of approximation schemes for fully nonlinear second order equations."

- Result: finite difference scheme converges to unique viscosity solution under three conditions

   1. Monotonicity.
   2. Consistency.
   3. Stability.

- Good reference: Tourin (2013), *An Introduction to Finite Difference Methods for PDEs in Finance*.

Viscosity solutions, I

- Relevant notion of "solutions" to HJB introduced by Pierre-Louis Lions and Michael G. Crandall in 1983 in the context of PDEs.
- Classical solution of a PDE:
$$F(x, u, Du, D^2u) = 0$$
is a function u in Ω that is continuous and differentiable that satisfies the PDE above.
- We want a weaker class of solutions than classical solutions.
- More concretely, we want to allow for points of non-differentiability of u (in this case, $V(a)$).
- Similarly, we want to allow for convex kinks in the value function $V(a)$.
- Different classes of "weaker solutions."

Viscosity solutions, II

- Subsolution: An upper semicontinuous function u in Ω is a "subsolution" of a PDE in the "viscosity sense" if for any point $x_0 \in \Omega$ and any C^2 function ϕ such that $\phi(x_0) = u(x_0)$ and $\phi \geq u$ in a neighborhood of x_0, we have:

$$F(x_0, \phi(x_0), D\phi(x_0), D^2\phi(x_0)) \leq 0$$

- Supersolution: A lower semicontinuous function u in Ω is defined to be a "supersolution" of a PDE in the "viscosity sense" if for any point $x_0 \in \Omega$ and any C^2 function ϕ such that $\phi(x_0) = u(x_0)$ and $\phi \leq u$ in a neighborhood of x_0, we have:

$$F(x_0, \phi(x_0), D\phi(x_0), D^2\phi(x_0)) \geq 0$$

- Viscosity solution: A continuous function "u" is a "viscosity solution" of the PDE if it is both a supersolution and a subsolution.

Viscosity solutions, III

- Viscosity solution is unique.
- A baby example: consider the boundary value problem $F(u') = |u'| - 1 = 0$, on $(-1, 1)$ with boundary conditions $u(-1) = u(1) = 0$. The unique viscosity solution is the function $u(x) = 1 - |x|$.
- Coincides with solution to sequence problem.
- Numerical methods designed to find viscosity solutions.
- Check, for more background, *User's Guide to Viscosity Solutions of Second Order Partial Differential Equations* by Michael G. Crandall, Hitoshi Ishii, and Pierre-louis Lions.
- Also, *Controlled Markov Processes and Viscosity Solutions* by Wendell H. Fleming and Halil Mete Soner.

Concluding remarks

Summary

- Moll et al. (2021) suggest a convenient computational methodology for heterogenous household model in continuous time.
- Computations steps are not very different from the discrete-time approach, but the efficiency gain is enormous due to the particular algorithm.
- The discretization scheme makes it easy to get the stationary distribution out of the solution.
- On top of the computational contribution, some of the closed-form results sharply characterize the model's interesting features:
  - Individual saving policy near the borrowing constraint in a closed-form.
  - Time to binding constraint.
  - Closed-form wealth distribution under certain parametric assumptions.
- Transitional dynamics can also be obtained using the FD method.
- The idiosyncratic labor productivity process can be extended to the Ornstein-Uhlenbeck process, a continuous-time counterpart of the AR(1) process.