Appendix: For Online Publication

A Household problem

The representative household owns equity in all firms, supplies labor $l_{H,t}$, and chooses consumption c_t to maximize lifetime utility:

$$\max_{\{c_t, l_{H,t}\}_{t=0}^{\infty}} \quad \mathbb{E}_0 \sum_{t=0}^{\infty} \beta^t u(c_t, l_{H,t}) \tag{A.1}$$

subject to the budget constraint:

$$c_t = w(S_t) l_{H,t} + \int D(s; S_t) \, d\Phi_t \tag{A.2}$$

where $D(s; S_t) := \Pi(zZk; S_t) - I(s; S_t) - \frac{\mu}{2} I(s; S_t)^2 - \xi \mathbb{I}\{I(s; S_t) \neq 0\}$ denotes the dividend paid by a firm with individual state $s = (k, z)$, and $w(S_t)$ is the competitive wage.

The household's first-order conditions yield the labor supply schedule and the stochastic discount factor:

$$w(S_t) = -\frac{u_{l_H}(c_t, l_{H,t})}{u_c(c_t, l_{H,t})} \tag{A.3}$$

$$\Xi(S_t, S_{t+1}) = \beta \frac{u_c(c_{t+1}, l_{H,t+1})}{u_c(c_t, l_{H,t})}. \tag{A.4}$$

The stochastic discount factor $\Xi(S, S')$ enters the firm's Bellman equation (Section 2 of the main text) through the discounting of future value.

For the fully analytical characterization in Section 4.2 of the main text, we specialize to GHH preferences $u(c, l_H) = \log(c - \eta l_H)$ with risk neutrality ($\sigma \to 0$), which yields $w(S) = \eta$ (constant) and $\Xi(S, S') = \beta$. This eliminates wage-driven general equilibrium feedback and delivers closed-form expressions for all equilibrium objects.

B Proofs of theoretical results

Theorem A.1 (Marginal benefit)**.**
The marginal benefit of investment $q(k,z;S) := \frac{\partial \mathbb{E}_{z,S}\Xi(S,S')J(k,z';S')}{\partial k}$ *satisfies*

$$q(k,z;S) = \frac{\partial \mathbb{E}_{z,S}\Xi(S,S')J^{NoFixed}(k,z';S')}{\partial k} = \mathcal{G}(S)z + \mathcal{H}(S) \tag{A.5}$$

for some $\mathcal{G},\mathcal{H} : \mathbb{S} \to \mathbb{R}$*, where* $\mathbb{S}$ *is the set of all possible aggregate states.*

Proof.
We provide two proofs. The first constructs the marginal benefit directly as an infinite sum of discounted future marginal profits. The second differentiates the Bellman equation and uses a guess-and-verify argument.

Proof 1: direct summation.
Consider a firm at date $\tau = 0$ with state (k_0, z_0) and aggregate state S_0. A marginal increase in investment I_0 raises the firm's capital stock by one unit at $\tau = 1$ and, after depreciation, by $(1-\delta)^{\tau-1}$ units at each future date $\tau \geq 1$. By CRS, the marginal profit from an additional unit of capital in period τ is $\Pi(1; S_\tau)z_\tau Z_\tau$, independent of k_τ. The fixed cost ξ is a lump sum paid per adjustment event and does not vary with the level of investment; the Poisson rate φ governs only whether adjustment occurs. Since neither k_0, ξ, nor φ enters the per-unit marginal return to capital, the marginal benefit of investment coincides with that of the frictionless problem $J^{NoFixed}$, establishing the first equality. The first-order condition equates the marginal cost $1 + \mu I_0$ to this marginal benefit:

$$\underbrace{1 + \mu I_0}_{\text{Marginal cost}} = \underbrace{\mathbb{E}_{z,S} \sum_{\tau=1}^{\infty} \left[\beta^\tau \left(\frac{u'(c_\tau)}{u'(c_0)} \right) \overbrace{(1-\delta)^{\tau-1}\Pi(1; S_\tau)z_\tau Z_\tau}^{\text{marginal profit}} \right]}_{\text{Marginal benefit in int. (\textbf{independent} of } I_0 \text{ and } k_0)} \tag{A.6}$$

Since $z_\tau = \rho^\tau z_0 + \sum_{j=1}^{\tau} \rho^{\tau-j}\epsilon_j$ with $\mathbb{E}[\epsilon_j] = \bar{\epsilon}$, each z_τ is affine in z_0. Collecting terms proportional to z_0 and the remainder:

$$= \left(\rho\beta\mathbb{E}_S\left(\frac{u'(c_1)}{u'(c_0)}\right)\pi'Z' + \rho^2\beta^2(1-\delta)\mathbb{E}_S\left(\frac{u'(c_2)}{u'(c_0)}\right)\pi''Z'' + \dots\right)z_0 \tag{A.7}$$

$$+ \left(\bar{\epsilon}\beta\mathbb{E}_S\left(\frac{u'(c_1)}{u'(c_0)}\right)\pi'Z' + \bar{\epsilon}(\rho+1)\beta^2(1-\delta)\mathbb{E}_S\left(\frac{u'(c_2)}{u'(c_0)}\right)\pi''Z'' + \dots\right) \tag{A.8}$$

$$= \underbrace{\mathcal{G}(S_0)z_0 + \mathcal{H}(S_0)}_{\text{Affine in } z_0} =: A(z_0; S_0). \tag{A.9}$$

The coefficient of z_0 defines $\mathcal{G}(S_0)$ and the constant defines $\mathcal{H}(S_0)$. Both series converge under the transversality condition $\beta(1-\delta) < 1$, and the recursive representations for $\mathcal{G}$ and $\mathcal{H}$ in the main text follow by splitting each sum at $\tau = 1$. Thus,

$$I_0 = \frac{A(z_0; S_0) - 1}{\mu} \tag{A.10}$$

■

Proof 2: Bellman equation (guess and verify).

The proof proceeds in two guess-and-verify steps.

Step 1 (Envelope condition: J_k is independent of k).

Guess. $J_k(k, z; S) = J_k(z; S)$, i.e., the marginal value of capital does not depend on the capital stock itself.

Verify. Differentiate the Bellman equation with respect to k. CRS gives $\frac{\partial}{\partial k}\Pi(zZk; S) = \Pi(1; S)zZ$, independent of k. For the continuation value, the envelope theorem applied to the optimal investment choice I^* gives:

$$\frac{\partial R^*}{\partial k} = (1-\delta)\,\mathbb{E}_{z,S}\Xi(S, S')\,J_k\big(k(1-\delta)+I^*, z'; S'\big),$$

$$\frac{\partial R^c}{\partial k} = (1-\delta)\,\mathbb{E}_{z,S}\Xi(S,S')\,J_k\big(k(1-\delta), z'; S'\big)\,.$$

Under the guess, the first arguments $k(1-\delta)+I^*$ and $k(1-\delta)$ are irrelevant, so $\frac{\partial R^*}{\partial k} = \frac{\partial R^c}{\partial k}$. Consequently, the max operator (adjust or not) and the Poisson probability φ drop out of J_k:

$$J_k(z;S) = \Pi(1;S)\,zZ + (1-\delta)\,\mathbb{E}_{z,S}\Xi(S,S')\,J_k(z';S'). \tag{A.11}$$

The right-hand side involves only (z,S), confirming the guess. Moreover, the frictionless value $J^{NoFixed}$ satisfies the identical recursion (with no ξ or φ), so $J_k = J_k^{NoFixed}$ by uniqueness of the fixed point.

Step 2 (Affine fixed point: q is affine in z).

Guess. $q(z;S) := \mathbb{E}_{z,S}\Xi(S,S')\,J_k(z';S') = \mathcal{G}(S)z + \mathcal{H}(S)$ for some functions $\mathcal{G},\mathcal{H} : \mathbb{S} \to \mathbb{R}$.

Verify. Substituting (A.11) into the definition of q:

$$q(z;S) = \mathbb{E}_{z,S}\Xi(S,S')\big[\Pi(1;S')z'Z' + (1-\delta)\,q(z';S')\big]\,.$$

Imposing the guess on the right-hand side and expanding $z' = \rho z + \epsilon'$ with $\mathbb{E}\epsilon' = \bar{\epsilon}$:

$$\begin{aligned}
q(z;S) &= \mathbb{E}_S\Xi(S,S')\big[\Pi(1;S')(\rho z+\epsilon')Z' + (1-\delta)\,\big(\mathcal{G}(S')(\rho z+\epsilon') + \mathcal{H}(S')\big)\big] \\
&= \underbrace{\mathbb{E}_S\big[\rho\,\Xi(S,S')\big(\Pi(1;S')Z' + (1-\delta)\,\mathcal{G}(S')\big)\big]}_{\text{coefficient of } z}\cdot z \\
&\quad + \underbrace{\bar{\epsilon}\,\mathbb{E}_S\big[\Xi(S,S')\big(\Pi(1;S')Z' + (1-\delta)\,\mathcal{G}(S')\big)\big] + \mathbb{E}_S\big[\Xi(S,S')\,(1-\delta)\,\mathcal{H}(S')\big]}_{\text{constant}}.
\end{aligned}$$

Matching the coefficient of z with $\mathcal{G}(S)$ and the constant with $\mathcal{H}(S)$:

$$\mathcal{G}(S) = \mathbb{E}\big[\rho\,\Xi(S,S')\big(\Pi(1;S')Z' + (1-\delta)\,\mathcal{G}(S')\big)\big]\,,$$

$$\mathcal{H}(S) = \tfrac{\bar{\epsilon}}{\rho}\,\mathcal{G}(S) + \mathbb{E}\left[\Xi(S,S')\,(1-\delta)\,\mathcal{H}(S')\right] = \left(\tfrac{1}{\rho}-1\right)\mathcal{G}(S) + \mathbb{E}\left[\Xi(S,S')\,(1-\delta)\,\mathcal{H}(S')\right],$$

where the last equality uses $\bar{\epsilon}/\rho = (1-\rho)/\rho = 1/\rho - 1$. Both are well-defined recursive equations in $\mathcal{G}$ and $\mathcal{H}$, confirming the affine guess. ■

Corollary A.1 (Forward-looking affine investment in intensive margin)**.**
Firm-level optimal investment conditional on adjustment is:

$$I^*(z;S) = \frac{\mathcal{G}(S)z + \mathcal{H}(S) - 1}{\mu}$$

Proof.
From the Marginal Benefit Theorem, the marginal benefit of investment is $A(z;S) = \mathcal{G}(S)z + \mathcal{H}(S)$, independent of k and I. The marginal cost is $1 + \mu I$ from the quadratic adjustment cost. Equating:

$$\mathcal{G}(S)z + \mathcal{H}(S) = 1 + \mu I^*(z;S).$$

Solving for I^* yields the result. ■

Theorem A.2 (The inaction region of productivity)**.**
A firm does not invest when z lies in the inaction interval $\Omega := [z_L, z_H]$ where:

$$\Omega(S) := [z_L(S), z_H(S)] = \left[\frac{1-\sqrt{2\mu\xi}-\mathcal{H}(S)}{\mathcal{G}(S)}, \frac{1+\sqrt{2\mu\xi}-\mathcal{H}(S)}{\mathcal{G}(S)}\right] \tag{A.12}$$

Proof.
A firm invests only when

$$\overbrace{\underbrace{(\mathcal{G}(S)z+\mathcal{H}(S))}_{\text{marginal lifetime profit per } k} \times \underbrace{I^*(z;S)}_{k \text{ to be installed}}}^{\text{Benefit in extensive margin. (independent of } k)} > \overbrace{-\xi - I^*(z;S) - \frac{\mu}{2}I^*(z;S)^2}^{\text{Cost in extensive margin}} \tag{A.13}$$

Since $I^*(z;S) = \frac{\mathcal{G}(S)z+\mathcal{H}(S)-1}{\mu}$, indifference gives a *quadratic* equation in the threshold $\bar{z}$:

$$(\mathcal{G}(S)\bar{z} + \mathcal{H}(S) - 1)^2 = 2\mu\xi \tag{A.14}$$

where *any z between the two thresholds* implies that (A.13) is not satisfied. The solution of the quadratic equation is as follows:

$$\bar{z} = \frac{1 \pm \sqrt{2\mu\xi} - \mathcal{H}(S)}{\mathcal{G}(S)} \tag{A.15}$$

Therefore,

$$[z_L(S), z_H(S)] = \left[\frac{1 - \sqrt{2\mu\xi} - \mathcal{H}(S)}{\mathcal{G}(S)}, \frac{1 + \sqrt{2\mu\xi} - \mathcal{H}(S)}{\mathcal{G}(S)}\right] \tag{A.16}$$

■

Corollary A.2 (The inaction midpoint)**.**
The midpoint of the inaction band z_M can be characterized as $z_M(S) = \frac{1-\mathcal{H}(S)}{\mathcal{G}(S)}$, and the unconstrained optimal investment at the midpoint is zero: $I^*(z_M;S) = 0$.

Proof.
From Theorem A.2:

$$z_M(S) = \frac{z_L(S) + z_H(S)}{2} = \frac{1}{2}\frac{(1 - \sqrt{2\mu\xi} - \mathcal{H}) + (1 + \sqrt{2\mu\xi} - \mathcal{H})}{\mathcal{G}(S)} = \frac{1 - \mathcal{H}(S)}{\mathcal{G}(S)}.$$

Evaluating I^* at z_M:

$$I^*(z_M;S) = \frac{\mathcal{G}(S)\frac{1-\mathcal{H}(S)}{\mathcal{G}(S)} + \mathcal{H}(S) - 1}{\mu} = \frac{1 - \mathcal{H}(S) + \mathcal{H}(S) - 1}{\mu} = 0.$$

■

Proposition A.2 (Aggregate-level selection effect: quality margin)**.**

If a shift in the aggregate productivity shock leads to $\Delta\mathcal{G}(S) > 0$ and $\Delta\mathcal{H}(S) > 0$, then the inaction midpoint decreases.

Proof.

From Corollary A.2, $z_M(S) = \frac{1-\mathcal{H}(S)}{\mathcal{G}(S)}$. The partial derivatives are:

$$\frac{\partial z_M}{\partial \mathcal{G}} = -\frac{1-\mathcal{H}(S)}{\mathcal{G}(S)^2}, \qquad \frac{\partial z_M}{\partial \mathcal{H}} = -\frac{1}{\mathcal{G}(S)}.$$

Under $\mathcal{G}(S) > 0$ (positive slope of marginal benefit), we have $\frac{\partial z_M}{\partial \mathcal{H}} < 0$ directly. For $\frac{\partial z_M}{\partial \mathcal{G}}$: the maintained condition $\mathcal{H}(S) < 1$ (the intercept of the marginal benefit is below unity at the steady state, reflecting that the average firm requires positive productivity to justify investment) gives $1 - \mathcal{H}(S) > 0$, hence $\frac{\partial z_M}{\partial \mathcal{G}} < 0$. Therefore, $\Delta\mathcal{G} > 0$ and $\Delta\mathcal{H} > 0$ jointly decrease z_M. ■

Proposition A.3 (Aggregate investment characterization)**.**
The frictionless and frictional aggregate investments $(I^{NoFixed}, I)$ are

$$I^{NoFixed}(S) = \frac{1-\varphi}{\mu}\mathcal{G}(S)\left(\frac{\bar{\epsilon}}{1-\rho} - z_M(S)\right), \quad I(S) = I^{NoFixed}(S) - \Delta_I(S),$$

where $\Delta_I(S) = \frac{1-\varphi}{\mu}\left(\mathcal{G}(S)M_1(S) + \left(\mathcal{H}(S) - 1\right)M_0(S)\right).$

Proof.

Step 1: Frictionless aggregate investment. In the frictionless economy ($\xi = 0$), all firms with $\varphi_{it} = 0$ (probability $1-\varphi$) invest at the unconstrained optimum:

$$I^{NoFixed}(S) = (1-\varphi)\int I^*(z;S)d\Phi_z = \frac{1-\varphi}{\mu}\left(\mathcal{G}(S)\frac{\bar{\epsilon}}{1-\rho} + \mathcal{H}(S) - 1\right),$$

where $\mathbb{E}[z] = \frac{\bar{\epsilon}}{1-\rho}$ is the stationary mean. Substituting $z_M(S) = \frac{1-\mathcal{H}(S)}{\mathcal{G}(S)}$:

$$I^{NoFixed}(S) = \frac{1-\varphi}{\mu}\mathcal{G}(S)\left(\frac{\bar{\epsilon}}{1-\rho} - z_M(S)\right).$$

Step 2: Frictional aggregate investment. With fixed costs, firms in the inaction band $\Omega(S)$ invest zero:

$$I(S) = (1-\varphi)\int_{z\notin\Omega(S)} I^*(z;S)d\Phi_z = I^{NoFixed}(S) - \frac{1-\varphi}{\mu}\int_{\Omega(S)} \left(\mathcal{G}(S)z + \mathcal{H}(S) - 1\right) d\Phi_z.$$

Step 3: Investment gap. Using $M_n(S) := \int_{\Omega(S)} z^n d\Phi_z$:

$$\Delta_I(S) := I^{NoFixed}(S) - I(S) = \frac{1-\varphi}{\mu}\left(\mathcal{G}(S)M_1(S) + (\mathcal{H}(S)-1)M_0(S)\right).$$

■

Proposition A.4 (Output gap characterization)**.**
The output gap Δ_Y is characterized as $\Delta_Y(S_{-1}, S) = O(S)Z(\rho\Delta_{II}(S_{-1}) + \bar{\epsilon}\Delta_I(S_{-1}))$.

Proof.
Under CRS production, aggregate output takes the form $Y = O(S)Z\int zk\, d\Phi$, where $O(S) = \Pi(1;S) - w(S)\Pi_w(1;S)$ is the net-of-labor productivity.

Step 1: Output gap structure. The output gap equals $\Delta_Y = Y^{NoFixed} - Y = O(S)Z\int z[I^*(z_{-1}) - \widehat{I}(z_{-1})]d\Phi_{-1}$, where the difference in capital stocks at the current date reflects last period's investment decisions, and $\widehat{I}$ denotes frictional investment ($= I^*$ outside Ω, $= 0$ inside Ω).

Step 2: Cross-sectional projection. Since firm productivity evolves as $z = \rho z_{-1} + \epsilon$ with ϵ independent of z_{-1}:

$$\int z\mathbb{I}\{z_{-1}\in\Omega\}I^*(z_{-1})d\Phi_{-1} = \rho\int_\Omega z_{-1}I^*(z_{-1})d\Phi_{-1} + \bar{\epsilon}\int_\Omega I^*(z_{-1})d\Phi_{-1}.$$

Step 3: Expressing via gap moments. Using $I^*(z_{-1}) = \frac{1}{\mu}(\mathcal{G}z_{-1} + \mathcal{H} - 1)$:

$$\int_\Omega z_{-1}^n I^*(z_{-1})d\Phi_{-1} = \frac{1}{\mu}(\mathcal{G}M_{n+1} + (\mathcal{H}-1)M_n).$$

Scaling by $(1-\varphi)$ and substituting into Step 2:

$$\Delta_Y = O(S)Z\left(\rho\underbrace{\frac{1-\varphi}{\mu}\left(\mathcal{G}M_2+(\mathcal{H}-1)M_1\right)}_{=:\Delta_{II}}+\bar{\epsilon}\underbrace{\frac{1-\varphi}{\mu}\left(\mathcal{G}M_1+(\mathcal{H}-1)M_0\right)}_{=\Delta_I}\right).$$

■

Proposition A.5 (Analytical equilibrium characterization)**.**

Given $\mathcal{G}$, $\mathcal{H}$, $\sigma_\epsilon \geq 0$, and $\xi \geq 0$, the RCE allocations are completely characterized by a vector of aggregate variables $S = [K, cov(k,z), Z]$.

Proof.

We verify that all equilibrium objects are functions of $S = [K, cov(k,z), Z]$ given $(\mathcal{G}, \mathcal{H})$.

(i) Firm-level policies. $I^*(z;S)$ depends only on $(\mathcal{G}(S), \mathcal{H}(S), z)$ (Corollary A.1). The inaction band $\Omega(S) = [z_L(S), z_H(S)]$ depends only on $(\mathcal{G}(S), \mathcal{H}(S), \xi, \mu)$ (Theorem A.2).

(ii) Inaction-band moments. $M_n(S) = \int_{\Omega(S)} z^n d\Phi_z$ depends on z_L, z_H (determined by $\mathcal{G}, \mathcal{H}$) and the time-invariant stationary distribution Φ_z (determined by $\rho, \bar{\epsilon}, \sigma_\epsilon$). Hence M_0, M_1, M_2 are functions of $(\mathcal{G}, \mathcal{H})$ only.

(iii) Aggregate quantities. From Proposition A.3, I and $I^{NoFixed}$ are functions of $(\mathcal{G}, \mathcal{H}, M_0, M_1)$. Output is $Y = O(S)Z(\frac{\bar{\epsilon}}{1-\rho}K + cov(k,z))$, which depends on $(K, cov(k,z), Z)$ and $O(S)$. Consumption $C = Y - I - AC$ is determined given the preceding objects.

(iv) Laws of motion.

$$K' = (1-\delta)K + I(S),$$

$$cov(k,z)' = \rho(1-\delta)cov(k,z) + \rho\left[\frac{1-\varphi}{\mu}\frac{\sigma_\epsilon^2}{1-\rho^2}\mathcal{G} + \Delta_I - \Delta_{II}\right],$$

$$Z' = \rho_{agg}Z + \epsilon_{agg},$$

are functions of $(K, cov(k,z), Z)$ and the equilibrium objects derived above. Since all quantities and their transitions are pinned down by S, the state S is sufficient.

■

Proposition A.6 (Analytical gaps)**.**
Under Gaussian productivity, the truncated moments M_0, M_1, M_2 admit closed-form expressions. In particular, the investment gap reduces to:

$$\Delta_I = I^{NoFixed}\left(\Phi^{SN}(z_H^*(S)) - \Phi^{SN}(z_L^*(S)) + \frac{\phi^{SN}(z_H^*(S)) - \phi^{SN}(z_L^*(S))}{z_M^*(S)}\right) \quad \text{(A.17)}$$

where $z_j^* := \frac{z_j - \frac{\bar{\epsilon}}{1-\rho}}{\sqrt{\frac{\sigma_\epsilon^2}{1-\rho^2}}}$, $j \in \{H, M, L\}$ is the normalized stochastic variable. ϕ^{SN} and Φ^{SN} are the probability density function and cumulative distribution function of the standard normal distribution.

Proof.
Part 1: Investment gap Δ_I.

From the prior derivations, we have

$$\Delta_I = \frac{1-\varphi}{\mu}\mathcal{G}(S)\left(M_1(S) - z_M(S)M_0(S)\right) \quad \text{(A.18)}$$

$$I^{NoFixed} = \frac{1-\varphi}{\mu}\mathcal{G}(S)\left(\frac{\bar{\epsilon}}{1-\rho} - z_M(S)\right) = -\frac{1-\varphi}{\mu}\mathcal{G}(S)\sigma_z z_M^*. \quad \text{(A.19)}$$

Combining these two, we have

$$\frac{\Delta_I}{I^{NoFixed}} = -\frac{1}{\sigma_z}\frac{1}{z_M^*}\left(M_1(S) - z_M(S)M_0(S)\right). \quad \text{(A.20)}$$

Using the standard identity $\int_{-\infty}^{z} y d\Phi = \mu_z\Phi^{SN}(z^*) - \sigma_z\phi^{SN}(z^*)$, where $z^* = (z-\mu_z)/\sigma_z$, $\mu_z = \frac{\bar{\epsilon}}{1-\rho}$, and $\sigma_z = \sqrt{\frac{\sigma_\epsilon^2}{1-\rho^2}}$:

$$M_1(S) = \int_{z_L}^{z_H} z d\Phi = \int_{-\infty}^{z_H} z d\Phi - \int_{-\infty}^{z_L} z d\Phi \quad \text{(A.21)}$$

$$= \frac{\bar{\epsilon}}{1-\rho}\left(\Phi^{SN}\left(z_H^*\right) - \Phi^{SN}\left(z_L^*\right)\right) - \sigma_z\left(\phi^{SN}\left(z_H^*\right) - \phi^{SN}\left(z_L^*\right)\right) \tag{A.22}$$

Also, we have

$$M_0(S) = \int_{z_L}^{z_H} d\Phi = \Phi^{SN}\left(z_H^*\right) - \Phi^{SN}\left(z_L^*\right). \tag{A.23}$$

Therefore,

$$M_1(S) - z_M(S)M_0(S) \tag{A.24}$$

$$= \left(\frac{\bar{\epsilon}}{1-\rho} - z_M\right)\left(\Phi^{SN}\left(z_H^*\right) - \Phi^{SN}\left(z_L^*\right)\right) - \sigma_z\left(\phi^{SN}\left(z_H^*\right) - \phi^{SN}\left(z_L^*\right)\right) \tag{A.25}$$

$$= -\sigma_z z_M^*\left(\Phi^{SN}\left(z_H^*\right) - \Phi^{SN}\left(z_L^*\right)\right) - \sigma_z\left(\phi^{SN}\left(z_H^*\right) - \phi^{SN}\left(z_L^*\right)\right). \tag{A.26}$$

Hence,

$$\frac{\Delta_I}{I^{NoFixed}} = -\frac{1}{\sigma_z}\frac{1}{z_M^*}\left(M_1(S) - z_M(S)M_0(S)\right) \tag{A.27}$$

$$= \Phi^{SN}(z_H^*(S)) - \Phi^{SN}(z_L^*(S)) + \frac{\phi^{SN}(z_H^*(S)) - \phi^{SN}(z_L^*(S))}{z_M^*(S)} \tag{A.28}$$

Part 2: Sorting gap Δ_{II} and output gap Δ_Y.

Step 1: Second moment M_2. We compute $\int_{-\infty}^{z} y^2 d\Phi$ using the standardized variable $y^* = (y-\mu_z)/\sigma_z$:

$$\int_{-\infty}^{z} y^2 d\Phi = \int_{-\infty}^{z^*} (\sigma_z y^* + \mu_z)^2 \phi^{SN}(y^*)dy^*$$
$$= \sigma_z^2 \int_{-\infty}^{z^*} (y^*)^2 \phi^{SN}(y^*)dy^* + 2\mu_z\sigma_z \int_{-\infty}^{z^*} y^*\phi^{SN}(y^*)dy^* + \mu_z^2\Phi^{SN}(z^*). \tag{A.29}$$

Using $\int_{-\infty}^{a}(y^*)^2\phi^{SN}(y^*)dy^* = \Phi^{SN}(a) - a\phi^{SN}(a)$ (integration by parts) and

$\int_{-\infty}^{a} y^* \phi^{SN}(y^*) dy^* = -\phi^{SN}(a)$:

$$\int_{-\infty}^{z} y^2 d\Phi = (\mu_z^2 + \sigma_z^2)\Phi^{SN}(z^*) - 2\mu_z\sigma_z\phi^{SN}(z^*) - \sigma_z^2 z^*\phi^{SN}(z^*). \tag{A.30}$$

Hence,

$$\begin{aligned} M_2(S) &= \int_{z_L}^{z_H} z^2 d\Phi \\ &= (\mu_z^2 + \sigma_z^2)(\Phi^{SN}(z_H^*) - \Phi^{SN}(z_L^*)) + 2\mu_z\sigma_z(\phi^{SN}(z_L^*) - \phi^{SN}(z_H^*)) \\ &\quad + \sigma_z^2(z_L^*\phi^{SN}(z_L^*) - z_H^*\phi^{SN}(z_H^*)). \end{aligned} \tag{A.31}$$

Step 2: Sorting gap Δ_{II}. From the definition, $\Delta_{II} = \frac{1-\varphi}{\mu}(\mathcal{G}M_2 + (\mathcal{H}-1)M_1)$. Substituting the expressions for M_1 and M_2, factoring through $I^{NoFixed} = -\frac{1-\varphi}{\mu}\mathcal{G}\sigma_z z_M^*$, and collecting terms yields the stated formula.

Step 3: Output gap Δ_Y. From Proposition A.4, $\Delta_Y(S_{-1}, S) = O(S)Z(\rho\Delta_{II}(S_{-1}) + \bar{\epsilon}\Delta_I(S_{-1}))$. Substituting the analytical forms and noting that the persistence-weighted contribution introduces a factor $\rho\sigma_z$ in place of σ_z in the sorting-gap terms yields the stated output gap expression. ∎

Lemma A.1 (Neutrality point)**.**
In the analytical equilibrium, there exists a level of aggregate productivity where the investment gap is zero:

$$Z^{Neutral} = \frac{1}{\rho_{agg}\bar{\epsilon}}\left(\frac{(1-\rho)(1-\rho_{agg}\beta(1-\delta))}{\alpha\left(\frac{1-\alpha}{\eta}\right)^{\frac{1-\alpha}{\alpha}}\beta} - \frac{\bar{\epsilon}_{agg}\bar{\epsilon}}{1-\beta(1-\delta)}\right)$$

Proof.
From the analytical investment gap, $\Delta_I = 0$ if and only if $z_M^* = 0$, which requires $z_M = \frac{\bar{\epsilon}}{1-\rho}$. Under linear utility (GHH with $\sigma \to 0$), $\mathcal{G}(Z)$ and $\mathcal{H}(Z)$ are affine in

Z with known coefficients. Setting $z_M(Z) = \frac{1-\mathcal{H}(Z)}{\mathcal{G}(Z)} = \frac{\bar{\epsilon}}{1-\rho}$, equivalently:

$$1 - \mathcal{H}(Z) = \mathcal{G}(Z)\frac{\bar{\epsilon}}{1-\rho}.$$

Substituting the explicit forms of $\mathcal{G}(Z)$ and $\mathcal{H}(Z)$, the equation becomes linear in Z. Solving yields the stated expression for $Z^{Neutral}$. ■

Lemma A.2 (Equivalence relation)**.**
The following equivalence holds: $z_M^*(Z) < 0 \iff Z > Z^{Neutral} \iff I^{NoFixed}(Z) > 0 \iff \Delta_I(Z) > 0$.

Proof.
(i) $z_M^* < 0 \iff Z > Z^{Neutral}$. From the proof of Lemma A.1, $z_M^* = 0 \iff Z = Z^{Neutral}$. Since $z_M(Z) = \frac{1-\mathcal{H}(Z)}{\mathcal{G}(Z)}$ is strictly decreasing in Z (the numerator $1-\mathcal{H}$ decreases while the denominator $\mathcal{G}$ increases in Z), z_M^* is strictly decreasing in Z, establishing the equivalence.

(ii) $z_M^* < 0 \iff I^{NoFixed} > 0$. $I^{NoFixed} = -\frac{1-\varphi}{\mu}\mathcal{G}\sigma_z z_M^*$. Since $\mathcal{G} > 0$ and $\sigma_z > 0$, the sign of $I^{NoFixed}$ is opposite to that of z_M^*.

(iii) $z_M^* < 0 \iff \Delta_I > 0$. From Proposition A.6, $\Delta_I / I^{NoFixed}$ involves the mass and asymmetry within the inaction band. When $z_M^* < 0$ (the band sits below the distribution mean), both the CDF-difference and density-ratio terms are positive, giving $\Delta_I / I^{NoFixed} > 0$. Combined with $I^{NoFixed} > 0$, this yields $\Delta_I > 0$. The converse follows by the same arguments. ■

Proposition A.7 (Analytical approximation to aggregate investment and output growth)**.**
Under linear utility, the analytical approximation is:

$$I(Z) = \underbrace{aZ + b}_{=I^{NoFixed}} - \underbrace{c(\phi^{SN})'\left(\frac{d}{Z+e} + f\right)}_{\approx \Delta_I} + O(\xi) \tag{A.32}$$

$$\Delta_Y(Z_{-1}, Z) = gZ\,\overbrace{c(\phi^{SN})'\left(\frac{d}{Z_{-1}+e}+f\right)}^{\approx \Delta_I(Z_{-1})} + O(\xi) \tag{A.33}$$

for known a, b, c, d, e, f, g functions of structural parameters, and where $(\phi^{SN})'$ is the first derivative of the standard normal p.d.f. and $O(\xi) \to 0$ at the same speed as $\xi \to 0$.

Proof.

Step 1: From Δ_I to the Gaussian derivative form. From the mean-value-theorem approximation developed in Appendix Section 1:

$$\Delta_I \approx \frac{(1-\varphi)2\sqrt{2\mu\xi}(1-\zeta_2)}{\zeta_1\mu}(\phi^{SN})'(\zeta_1 z_M^*). \tag{A.34}$$

Define $c := \frac{(1-\varphi)2\sqrt{2\mu\xi}(1-\zeta_2)}{\zeta_1\mu}$.

Step 2: Express $\zeta_1 z_M^*$ as a Möbius function of Z. Under linear utility, $\mathcal{G}(Z)$ and $\mathcal{H}(Z)$ are affine in Z. Write $\mathcal{G}(Z) = \alpha_1 Z + \alpha_2$ and $\mathcal{H}(Z) = \alpha_3 Z + \alpha_4$ with known coefficients. Then:

$$z_M(Z) = \frac{1-\mathcal{H}(Z)}{\mathcal{G}(Z)} = \frac{(1-\alpha_4)-\alpha_3 Z}{\alpha_1 Z + \alpha_2}, \tag{A.35}$$

a Möbius transformation. Hence $z_M^* = (z_M - \mu_z)/\sigma_z$ is also a Möbius transformation of Z, taking the form $\frac{pZ+q}{rZ+s}$ for constants determined by $(\alpha_1, \ldots, \alpha_4, \mu_z, \sigma_z)$.

Rearranging: $\zeta_1 z_M^*(Z) = \frac{d}{Z+e} + f$, where

$$e = \frac{\alpha_2}{\alpha_1} = \frac{\bar{\epsilon}_{agg}}{\rho_{agg}(1-\rho\beta(1-\delta))},$$

and d, f are the coefficients given in Appendix (full analytics), obtained by decomposing the Möbius form into a constant plus a simple fraction.

Step 3: Investment function. Combining $I = I^{NoFixed} - \Delta_I$ with $I^{NoFixed} = aZ + b$:

$$I(Z) = (aZ + b) - c(\phi^{SN})'\left(\frac{d}{Z+e} + f\right) + O(\xi), \tag{A.36}$$

where $O(\xi)$ accounts for the two mean-value-theorem approximation errors, each of order $(z_H^* - z_L^*)^2 = O(\xi)$.

Step 4: Output gap. From $\Delta_Y(S_{-1}, S) = O(S)Z(\rho\Delta_{II}(S_{-1}) + \bar{\epsilon}\Delta_I(S_{-1}))$ and the proportionality of Δ_{II} to Δ_I under the Gaussian derivative approximation (with a correction captured by ζ_3), define $g := O(S)(\rho\frac{\zeta_3}{1-\zeta_2}\sigma_z + \frac{\bar{\epsilon}}{1-\rho})$ to obtain:

$$\Delta_Y(Z_{-1}, Z) = gZ\, c(\phi^{SN})'\left(\frac{d}{Z_{-1}+e} + f\right) + O(\xi). \tag{A.37}$$

■

Investment neutrality under symmetry. From the investment gap derivation, we have

$$\Delta_I(S) = \frac{1-\varphi}{\mu}\mathcal{G}(S)M_0(S)\left(\frac{M_1(S)}{M_0(S)} - z_M(S)\right). \tag{A.38}$$

Assuming a non-trivial economy ($\mathcal{G}(S) \neq 0$, $M_0(S) \neq 0$), $\Delta_I = 0$ requires $M_1(S)/M_0(S) = z_M(S)$, i.e., the average productivity among inactive firms equals the inaction midpoint. This holds whenever Φ is symmetric around z_M on $[z_L, z_H]$, since symmetry forces the midpoint and the within-band mean to coincide. Combined with the linearity of $I^*(z; S)$ in z (Corollary A.1) and $I^*(z_M; S) = 0$ (Corollary A.2), positive and negative foregone investments cancel exactly.

Proposition A.8 (Impossibility of macroeconomic neutrality)**.**
For any aggregate state S with $M_0(S) > 0$ and $\Delta_I(S) \geq 0$ (equivalently, $I(S) \geq 0$):

(i) $\Delta_{II}(S) > 0$.

(ii) When $\Delta_I(S) > 0$, all three gaps are simultaneously strictly positive: $\Delta_I(S) > 0$,

$\Delta_{II}(S) > 0$, *and* $\Delta_Y(S, S') > 0$ *for any subsequent state* S'.

In particular, the investment and output gaps cannot vanish simultaneously.

Proof.

The proof proceeds in three steps.

Step 1: Jensen's hierarchy. Whenever the within-band distribution has positive variance (Φ is not a point mass on Ω), Jensen's inequality applied to the strictly convex function $f(x) = x^2$ yields

$$f\left(\int_\Omega z \frac{\phi(z)}{M_0} dz\right) < \int_\Omega f(z) \frac{\phi(z)}{M_0} dz, \tag{A.39}$$

i.e., $(M_1/M_0)^2 < M_2/M_0$, which is equivalent to

$$M_1(S)^2 < M_2(S)\, M_0(S). \tag{A.40}$$

Dividing both sides by $M_0 M_1 > 0$ yields the moment ranking $M_2/M_1 > M_1/M_0$.

Step 2: Joint positivity in the empirically relevant region. When $\Delta_I(S) \geq 0$, we have $M_1/M_0 \geq z_M$. Combined with the ranking $M_2/M_1 > M_1/M_0$ from Step 1, this gives $M_2/M_1 > z_M$, and hence $\Delta_{II}(S) > 0$. This proves Part (i). In the strict case $\Delta_I(S) > 0$, both gaps are strictly positive, so $\Delta_Y(S, S') = O(S')Z'(\rho\, \Delta_{II}(S) + \bar{\epsilon}\, \Delta_I(S)) > 0$ since $O(S') > 0$, $Z' > 0$, and both gap terms are positive. This proves Part (ii).

Step 3: Impossibility of joint neutrality. As a corollary, $\Delta_I(S) = 0$ implies $\Delta_{II}(S) > 0$ by Part (i), so $\Delta_Y(S, S') = O(S')Z'\rho\, \Delta_{II}(S) > 0$ when $\Delta_I(S) = 0$. Conversely, $\Delta_Y = 0$ requires $\rho\, \Delta_{II} + \bar{\epsilon}\, \Delta_I = 0$; combined with $\Delta_I = 0$ this forces $\Delta_{II} = 0$, contradicting Part (i). Therefore the investment and output gaps cannot vanish simultaneously.

The economic content is as follows. The moment ranking $M_2/M_1 > M_1/M_0$ reflects a fundamental asymmetry: high-productivity inactive firms are overrepresented in the sorting margin precisely because they are high-productivity. In the empirically relevant region (left-of-peak, $\Delta_I > 0$), both the level of aggregate investment and the efficiency of its allocation are simultaneously distorted. The same symmetry that could neutralize the investment gap leaves the sorting gap strictly positive. Since Δ_Y is a positive linear combination of both gaps, output neutrality is generically impossible. ■

Proposition A.9 (Firm-level investment semi-elasticity).
The firm-level investment semi-elasticity at the steady state takes the stated four cases.

Proof.
At the steady state, $\mathcal{G}^{ss}$ and $\mathcal{H}^{ss}$ are each proportional to $\beta = 1/R$.

Case 1: Firm stays unconstrained ($z \notin \Omega$ before and after).

$$\widehat{I}^{ss}(z) = \frac{\mathcal{G}^{ss}z + \mathcal{H}^{ss} - 1}{\mu}, \qquad \frac{\partial \widehat{I}^{ss}}{\partial R} = -\frac{1}{R}\frac{\mathcal{G}^{ss}z + \mathcal{H}^{ss}}{\mu} = -\frac{1 + \mu\widehat{I}^{ss}}{R\mu}. \tag{A.41}$$

Dividing by $\widehat{I}^{ss}$: $\frac{\partial \log(\widehat{I}^{ss})}{\partial R} = -\frac{1}{R}\left(1 + \frac{1}{\mu\widehat{I}^{ss}}\right)$.

Case 2: Firm becomes constrained (enters inaction band). The firm at the boundary z_H^{ss} switches from $I^*(z_H) = \sqrt{2\mu\xi}/\mu > 0$ to zero. The discrete jump yields elasticity -1 at the threshold.

Case 3: Firm stays constrained ($z \in \Omega$ before and after). $\widehat{I} = 0$ throughout; elasticity is identically zero.

Case 4: Firm becomes unconstrained (exits inaction band). The firm jumps from $\widehat{I} = 0$ to $I^*(z) > 0$, a discrete positive change from a base of zero: $\frac{\partial \log(\widehat{I}^{ss})}{\partial R} = +\infty$. ■

Proposition A.10 (Aggregate investment semi-elasticity).
The aggregate investment semi-elasticities at the steady state are as stated in the main text.

Proof.

Aggregate investment is $I^{ss} = (1-\varphi)\int_{z\notin\Omega} I^*(z)d\Phi$. Differentiating with respect to R using Leibniz's rule:

$$\frac{\partial I^{ss}}{\partial R} = (1-\varphi)\Bigg[\underbrace{\int_{z\notin\Omega}\frac{\partial I^*(z)}{\partial R}d\Phi}_{\text{intensive margin}} \underbrace{-I^*(z_H^{ss})\phi(z_H^{ss})\frac{\partial z_H^{ss}}{\partial R} + I^*(z_L^{ss})\phi(z_L^{ss})\frac{\partial z_L^{ss}}{\partial R}}_{\text{extensive margin}}\Bigg]. \quad \text{(A.42)}$$

Intensive margin. Since $\frac{\partial I^*(z)}{\partial R} = -\frac{1}{R\mu}(\mathcal{G}^{ss}z + \mathcal{H}^{ss})$:

$$(1-\varphi)\int_{z\notin\Omega}\frac{\partial I^*}{\partial R}d\Phi = \frac{I^{NoFixed,ss} + \frac{1-\varphi}{\mu}(1-M_0^{ss})}{-R} = \frac{I^{ss} + \Delta_I^{ss} + \frac{1-\varphi}{\mu}(1-M_0^{ss})}{-R}. \quad \text{(A.43)}$$

Dividing the full expression by I^{ss} and collecting terms:

$$\frac{\partial\log(I^{ss})}{\partial R} = \left(1+\frac{\Delta_I^{ss}}{I^{ss}}\right)\frac{\partial\log(I^{NoFixed,ss})}{\partial R} - \frac{1}{I^{ss}}\left(I^*(z_H^{ss})\phi(z_H^{ss})\frac{\partial z_H^{ss}}{\partial R} - I^*(z_L^{ss})\phi(z_L^{ss})\frac{\partial z_L^{ss}}{\partial R}\right). \quad \text{(A.44)}$$

For the frictionless benchmark, $I^{NoFixed,ss} = \frac{1-\varphi}{\mu}\left(\frac{\Pi^{ss}(1)\overline{Z\epsilon}}{R(1-\rho)(1-\beta(1-\delta))} - 1\right)$, so:

$$\frac{\partial\log(I^{NoFixed,ss})}{\partial R} = -\frac{1}{R}\left(1+\frac{1}{\frac{\Pi^{ss}(1)\overline{Z\epsilon}}{R(1-\rho)(1-\beta(1-\delta))}-1}\right). \quad \text{(A.45)}$$

■

Corollary A.3 (Elasticity amplification)**.**
If $\Delta_I^{ss} \geq 0$, the frictional aggregate investment semi-elasticity at the steady state is greater than the frictionless benchmark: $\left|\frac{\partial\log(I^{ss})}{\partial R}\right| > \left|\frac{\partial\log(I^{NoFixed,ss})}{\partial R}\right|$.

Proof.

From Proposition A.10:

$$\frac{\partial \log(I^{ss})}{\partial R} = \underbrace{\left(1 + \frac{\Delta_I^{ss}}{I^{ss}}\right) \frac{\partial \log(I^{NoFixed,ss})}{\partial R}}_{\text{amplified intensive margin}} - \underbrace{\frac{1}{I^{ss}} \frac{\partial \Delta_I^{ss}}{\partial R}}_{\text{extensive margin}} .$$

Intensive margin amplification. If $\Delta_I^{ss} \geq 0$, then $1 + \frac{\Delta_I^{ss}}{I^{ss}} \geq 1$. Since $\frac{\partial \log(I^{NoFixed,ss})}{\partial R} < 0$, the first term already exceeds $\left|\frac{\partial \log(I^{NoFixed,ss})}{\partial R}\right|$ in absolute value.

Extensive margin reinforcement. An increase in R widens the inaction band, so $\frac{\partial \Delta_I^{ss}}{\partial R} > 0$:

$$\frac{\partial \Delta_I^{ss}}{\partial R} = \underbrace{I^*(z_H^{ss})\phi(z_H^{ss})\frac{\partial z_H^{ss}}{\partial R}}_{>0} - \underbrace{I^*(z_L^{ss})\phi(z_L^{ss})\frac{\partial z_L^{ss}}{\partial R}}_{<0} > 0.$$

The first term is positive ($I^*(z_H) > 0$, $\frac{\partial z_H}{\partial R} > 0$). The second subtracted term is negative ($I^*(z_L) < 0$, $\frac{\partial z_L}{\partial R} > 0$). Both contribute positively.

Since this positive extensive-margin term is subtracted in the semi-elasticity (and $I^{ss} > 0$), it further increases the magnitude:

$$\left|\frac{\partial \log(I^{ss})}{\partial R}\right| > \left|\left(1 + \frac{\Delta_I^{ss}}{I^{ss}}\right) \frac{\partial \log(I^{NoFixed,ss})}{\partial R}\right| \geq \left|\frac{\partial \log(I^{NoFixed,ss})}{\partial R}\right|.$$

■

Proposition A.12 (Existence of the critical value)**.**
Suppose the unconditional mean of productivity lies outside the inaction band: $\mu_z = \frac{\bar{\epsilon}}{1-\rho} \notin [z_L, z_H]$.[1] *If* $\Delta_I > 0$ *for some* σ_ϵ*, then there exists* σ_ϵ^* *such that (i)* $\Delta_I(\sigma_\epsilon^*) \geq \Delta_I(\sigma_\epsilon)$ *for all* $\sigma_\epsilon \in (0, \infty)$*, and (ii)* $\frac{\partial \Delta_I}{\partial \sigma_\epsilon}\big|_{\sigma_\epsilon^*} = 0$.

Proof.

[1] This condition holds whenever the fixed cost is small enough that the half-width of the inaction band $\sqrt{2\mu\xi/\mathcal{G}}$ is less than $|\mu_z - z_M|$, i.e., the band is narrower than the distance from its center to the unconditional mean. This is satisfied at the calibration.

Step 1: Boundary behavior.

As $\sigma_\epsilon \to 0$: The stationary distribution degenerates to a point mass at $\mu_z = \frac{\bar{\epsilon}}{1-\rho}$. The inaction band $[z_L, z_H]$ has finite width $W = 2\sqrt{2\mu\xi}/\mathcal{G}$ independent of σ_ϵ, and its midpoint z_M is likewise independent of σ_ϵ.

By hypothesis, $\mu_z \notin [z_L, z_H]$. Without loss of generality suppose $\mu_z > z_H$ (the case $\mu_z < z_L$ is symmetric). The standardized thresholds are $z_H^* = (z_H - \mu_z)/\sigma_z$ and $z_L^* = (z_L - \mu_z)/\sigma_z$. Since $z_H < \mu_z$ and $z_L < \mu_z$, both z_H^* and z_L^* diverge to $-\infty$ as $\sigma_z \to 0$. Therefore $M_0 = \Phi^{SN}(z_H^*) - \Phi^{SN}(z_L^*) \to 0$, and hence $\Delta_I = \frac{1-\varphi}{\mu}\mathcal{G}(M_1 - z_M M_0) \to 0$.

Remark. If instead $\mu_z \in (z_L, z_H)$, the standardized thresholds diverge with opposite signs ($z_H^* \to +\infty$, $z_L^* \to -\infty$), so $M_0 \to 1$ and $\Delta_I \to \frac{1-\varphi}{\mu}\mathcal{G}(\mu_z - z_M) = I^{NoFixed} \neq 0$. In this regime, Δ_I does not vanish as $\sigma_\epsilon \to 0$; all firms are inactive and the gap equals the full frictionless investment. The extreme value theorem argument in Step 2 then does not guarantee an interior maximizer, which is why the condition $\mu_z \notin [z_L, z_H]$ is imposed.

As $\sigma_\epsilon \to \infty$: The distribution becomes flat. The inaction band has fixed width $2\sqrt{2\mu\xi}/\mathcal{G}$. As $\sigma_\epsilon \to \infty$, $z_H^* - z_L^* \to 0$ (the normalized band shrinks), so $M_0 = \Phi^{SN}(z_H^*) - \Phi^{SN}(z_L^*) \to 0$. Hence $\Delta_I \to 0$.

Step 2: Continuity and the extreme value theorem. $\Delta_I(\sigma_\epsilon)$ is continuous on $(0, \infty)$ (composed of smooth functions of σ_ϵ through the normal CDF and PDF). From Step 1, $\Delta_I \to 0$ at both boundaries. By hypothesis, $\Delta_I > 0$ at some interior point. On any compact interval $[\varepsilon, 1/\varepsilon]$ (for ε sufficiently small), Δ_I achieves its maximum by the extreme value theorem. Since boundary values are arbitrarily small, this maximum is interior. Hence there exists σ_ϵ^* with $\Delta_I(\sigma_\epsilon^*) \geq \Delta_I(\sigma_\epsilon)$ for all σ_ϵ, and the first-order condition $\frac{\partial \Delta_I}{\partial \sigma_\epsilon}\big|_{\sigma_\epsilon^*} = 0$ holds. ■

Lemma A.3 (Two channels of the uncertainty effect)**.**
The following equivalence holds: $\frac{\partial \Delta_I}{\partial \sigma_\epsilon} = 0 \iff \frac{\partial \log(M_0)}{\partial \sigma_\epsilon} = -\frac{\partial \log\left(\frac{M_1}{M_0} - z_M\right)}{\partial \sigma_\epsilon}$.

Proof.

Since $\mathcal{G}$ and z_M are independent of σ_ϵ, differentiating $\Delta_I = \frac{1-\varphi}{\mu}\mathcal{G}\, M_0 \left(\frac{M_1}{M_0} - z_M\right)$:

$$\frac{\partial \Delta_I}{\partial \sigma_\epsilon} = \frac{1-\varphi}{\mu}\mathcal{G}\left[\left(\frac{M_1}{M_0} - z_M\right)\frac{\partial M_0}{\partial \sigma_\epsilon} + M_0 \frac{\partial}{\partial \sigma_\epsilon}\left(\frac{M_1}{M_0} - z_M\right)\right]. \tag{A.46}$$

Setting this to zero and dividing by $\frac{1-\varphi}{\mu}\mathcal{G}\, M_0 \left(\frac{M_1}{M_0} - z_M\right) \neq 0$ (when $\Delta_I \neq 0$):

$$0 = \frac{\partial \log(M_0)}{\partial \sigma_\epsilon} + \frac{\partial \log\left(\frac{M_1}{M_0} - z_M\right)}{\partial \sigma_\epsilon}, \tag{A.47}$$

which rearranges to the stated equivalence. ■

Proposition A.13 (Asymmetric and state-dependent GIRF)**.**
The generalized impulse response of aggregate investment is asymmetric between positive and negative shocks and state-dependent:

(i) $|g(S, \Delta_{\mathcal{H}})| < |g(S, -\Delta_{\mathcal{H}})|$*, for inaction bands to the left of the peak of the firm-level productivity distribution.*

(ii) S affects g through the intensive margin: $\frac{1-\varphi}{\mu}\Delta_{\mathcal{H}}(1 - M_0(S, \Delta_{\mathcal{H}}))$ *decreasing in* M_0

(iii) S affects g through the extensive margin: $\Delta_I(S, \Delta_{\mathcal{H}}) - \Delta_I(S, 0)$.

Proof.
Part (i): Asymmetry. A shock $\Delta_{\mathcal{H}}$ shifts the marginal benefit intercept, moving the midpoint to $z_M - \Delta_{\mathcal{H}}/\mathcal{G}$. The entire inaction band shifts by $-\Delta_{\mathcal{H}}/\mathcal{G}$.

The GIRF decomposes as $g = g^{NoFixed} - (\Delta_I(S, \Delta_{\mathcal{H}}) - \Delta_I(S, 0))$, where $g^{NoFixed} = \frac{1-\varphi}{\mu}\Delta_{\mathcal{H}}$ is symmetric in $\Delta_{\mathcal{H}}$.

When the inaction band is left of the distribution peak ($z_M^* < 0$):

Negative shock ($-\Delta_{\mathcal{H}}$, with $\Delta_{\mathcal{H}} > 0$): The band shifts rightward toward the high-density peak. The inactive mass M_0 increases substantially, so $\Delta_I(S, -\Delta_{\mathcal{H}}) > \Delta_I(S, 0)$. The extensive-margin term $-(\Delta_I(S, -\Delta_{\mathcal{H}}) - \Delta_I(S, 0)) < 0$ reinforces the negative intensive margin, amplifying $|g|$.

Positive shock ($\Delta_{\mathcal{H}} > 0$): The band shifts further left into the lower-density tail. M_0 decreases modestly, so $\Delta_I(S, \Delta_{\mathcal{H}}) < \Delta_I(S, 0)$. The extensive-margin term $-(\Delta_I(S, \Delta_{\mathcal{H}}) - \Delta_I(S, 0)) > 0$ reinforces the positive intensive margin, but the reinforcement is quantitatively small because the band moves into the thin tail where density is low.

Formally, the asymmetry follows from the unimodality of ϕ^{SN}: with $z^*_M < 0$, the band is between the mode and the left tail. A rightward shift (negative shock) moves the band toward the mode, where ϕ^{SN} is higher, producing a large increase in M_0 and a large rise in Δ_I. A leftward shift (positive shock) moves the band away from the mode into the tail, where ϕ^{SN} is lower, producing only a modest decrease in M_0 and Δ_I. Since the extensive-margin reinforcement is larger under the negative shock than under the positive shock, $|g(S, -\Delta_{\mathcal{H}})| > |g(S, \Delta_{\mathcal{H}})|$.

Part (ii): Intensive margin state dependence. $\frac{1-\varphi}{\mu}\Delta_{\mathcal{H}}(1 - M_0(S, \Delta_{\mathcal{H}}))$ is decreasing in M_0. This is immediate from the definition.

Part (iii): Extensive margin state dependence. $\Delta_I(S, \Delta_{\mathcal{H}}) - \Delta_I(S, 0)$ depends on the density at the band boundaries, which varies with S as different aggregate states position the band at different points relative to the distribution. ■

Corollary A.4 (Symmetric and state-independent frictionless GIRF)**.**
(i) $|g^{NoFixed}(S, \Delta_{\mathcal{H}})| = |g^{NoFixed}(S, -\Delta_{\mathcal{H}})|$. *(ii)* $|g^{NoFixed}(S_1, \Delta_{\mathcal{H}})| = |g^{NoFixed}(S_0, \Delta_{\mathcal{H}})|$ *for all* S_0, S_1.

Proof.
$g^{NoFixed}(S, \Delta_{\mathcal{H}}) = I^{NoFixed}(S, \Delta_{\mathcal{H}}) - I^{NoFixed}(S, 0) = \frac{1-\varphi}{\mu}\Delta_{\mathcal{H}}$. This is (i) linear in $\Delta_{\mathcal{H}}$, giving symmetry; and (ii) independent of S, giving state independence. ■

Lemma A.4 (The lemma for the sign determination)**.**
Suppose a random variable x follows the standard normal distribution. For any differen-

tiable function $g : \mathbb{R} \to \mathbb{R}$ such that $g' < 0$, the following inequality holds:

$$\int_{-\infty}^{\infty} xg(x)\phi^{SN}(x)dx < 0.$$

Proof.

Define $\chi(x) := g(-x) - g(x)$ for all $x > 0$. Note that $g' < 0$ implies $\chi > 0$.

$$\begin{aligned}
\int_{-\infty}^{\infty} xg(x)\phi^{SN}(x)dx &= \int_{-\infty}^{0} xg(x)\phi^{SN}(x)dx + \int_{0}^{\infty} xg(x)\phi^{SN}(x)dx \\
&= \int_{0}^{\infty} (-x)g(-x)\phi^{SN}(-x)dx + \int_{0}^{\infty} xg(x)\phi^{SN}(x)dx \\
&= -\int_{0}^{\infty} x(\chi(x) + g(x))\phi^{SN}(x)dx + \int_{0}^{\infty} xg(x)\phi^{SN}(x)dx \\
&= -\int_{0}^{\infty} x\chi(x)\phi^{SN}(x)dx < 0.
\end{aligned}$$

■

Proposition A.14 (The nonlinearity in the business cycle)**.**
If $I^{NoFixed}(Z) > 0$ and $\mathbb{P}(Z < Z^{Max})$ is sufficiently small,

(i) $\mathbb{E}I(Z) < I(\mathbb{E}Z)$

(ii) $s.d.(I) > s.d.(I^{NoFixed})$

(iii) $skewness(I) < skewness(I^{NoFixed})$

Proof.

Part (i): Cost of business cycle.

Write $I(Z) = I^{NoFixed}(Z) - \Delta_I(Z)$. Since $I^{NoFixed}(Z) = aZ + b$ is affine in Z:

$$\mathbb{E}I(Z) - I(\mathbb{E}Z) = -\left(\mathbb{E}[\Delta_I(Z)] - \Delta_I(\mathbb{E}Z)\right). \tag{A.48}$$

Thus $\mathbb{E}I(Z) < I(\mathbb{E}Z) \iff \mathbb{E}[\Delta_I(Z)] > \Delta_I(\mathbb{E}Z)$, i.e., Δ_I is convex in Z on average.

From the analytical approximation, $\Delta_I(Z) \approx c(\phi^{SN})'(\frac{d}{Z+e} + f)$. The gap $\Delta_I(Z)$ is hump-shaped, peaking at Z^{Max} and vanishing in both tails. Under the maintained condition that $\mathbb{P}(Z < Z^{Max})$ is sufficiently small, the economy operates predominantly in the right tail, where $\Delta_I(Z)$ is decreasing and convex in Z (flattening toward zero). By Jensen's inequality applied to this locally convex function:

$$\mathbb{E}[\Delta_I(Z)] > \Delta_I(\mathbb{E}[Z]), \tag{A.49}$$

establishing $\mathbb{E}I(Z) < I(\mathbb{E}Z)$.

Part (ii): Excess volatility.

Decompose the variance:

$$Var(I) = Var(I^{NoFixed} - \Delta_I) = Var(I^{NoFixed}) + Var(\Delta_I) - 2Cov(I^{NoFixed}, \Delta_I). \tag{A.50}$$

Since $I^{NoFixed} = aZ + b$ with $a > 0$, $Cov(I^{NoFixed}, \Delta_I) = aCov(Z, \Delta_I)$.

Under the maintained conditions, $\Delta_I(Z)$ is predominantly decreasing in Z over the high-density region, so $Cov(Z, \Delta_I) < 0$ and $Cov(I^{NoFixed}, \Delta_I) < 0$. Therefore:

$$Var(I) = Var(I^{NoFixed}) + \underbrace{Var(\Delta_I)}_{>0} + \underbrace{2|Cov(I^{NoFixed}, \Delta_I)|}_{>0} > Var(I^{NoFixed}). \tag{A.51}$$

Taking square roots: $s.d.(I) > s.d.(I^{NoFixed})$.

Part (iii): Negative skewness.

$I^{NoFixed}(Z) = aZ + b$ inherits the distributional shape of Z: if Z is symmetric (Gaussian), then $skewness(I^{NoFixed}) = 0$.

Since $\Delta_I(Z)$ is convex over the predominant support, $I(Z) = I^{NoFixed}(Z) - \Delta_I(Z)$ is concave there. A concave transformation of a symmetric random variable generates negative skewness: positive shocks to Z produce smaller-than-linear increases in I while negative shocks produce larger-than-linear decreases, creating a heavier left tail.

Formally, for a concave function f and symmetric random variable Z, $\mathbb{E}[(f(Z) - \mathbb{E}f(Z))^3] < \mathbb{E}[(aZ + b - \mathbb{E}[aZ + b])^3]$ (the standard result that concave transformations reduce skewness). Hence $skewness(I) < skewness(I^{NoFixed})$. ■